Proof of Reimer's Theorem

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Abstract

Two events in a product space $A, B \subset \Omega = \Omega_1 \times \dots \Omega_n$ are said to occur disjointly if we can *observe* them occurring on disjoint sets of indices $J, K \subset [n]$. We denote the space of configurations in which A and B occur disjointly by $A \Box B$. Reimer's theorem states that as long as all the sets Ω_i are finite, we have the inequality $\mathbb{P}(A \Box B) \leq \mathbb{P}(A)\mathbb{P}(B)$.

We explain Reimer's proof of the theorem [4], and mention some alterations introduced in Borgs, Chayes, and Randall's proof [3].

1 Introduction

Suppose that there are two friends, Yorick and Ren, each of whom wants to get a date to the dance on Friday. They each have a list of people with whom they are willing to go with. Then if we denote by $\mathbb{P}(Y)$ the odds that one of Yorick's date-candidates is available for the night, and $\mathbb{P}(R)$ the probability that one of Ren's date-candidates is available for the night, then the box product $\mathbb{P}(R \Box Y)$denotes the odds that both of them will be able to find *different* dates, and so both of them will be able to attend. Reimer's theorem tells us that $\mathbb{P}(R \Box Y) \leq \mathbb{P}(R)\mathbb{P}(Y)$.

2 Preliminaries

Let Ω_i be a finite set for each $i \in [n]$. If $\omega \in \Omega = \prod_{i=1}^n \Omega_i$, and if $K \subset [n]$, we define the *K-cylinder about* ω, $C(K, \omega)$, to be the set of all $\omega' \in \Omega$ which satisfy $\omega'_i = \omega_i$ for all $i \in K$. If $A, B \subset \Omega$, and $\omega \in A \cap B$, we say that $\omega \in A \Box B$ if we can produce disjoint index sets, $K, J \subset [n]$, such that $C(K, \omega) \subset A$ and $C(J, \omega) \subset B$. This captures the notion of disjoint occurrence: we know that A occurs on the indices in K and B occurs on the indices in J, and the two are not allowed to help one another.

The Van den Berg-Kesten inequality (BK) states that with the above hypotheses (i.e. finiteness of the Ω_i), we have

$$\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B).$$

3 The van den Berg-Fiebig Theorem

Reimer's theorem is a statement which applies to all finite probability spaces–the distribution functions are irrelevant. Yet the proof of Reimer's theorem is concentrated in some very clever discrete counting arguments, which would seem to only apply to finite or at least discrete measure spaces. The van den Berg-Fiebig theorem tells us that it is enough to prove the BK inequality for some of the simplest nontrivial product spaces: $\Omega = \{0,1\}^n$ endowed with the uniform probability. We follow the proof given in [1].

Proposition 3.1. *The BK conjecture holds for all product spaces* $\prod_{i=1}^n \Omega_i$ *if and only if it holds in the case where each* Ω_i *is* $\{0,1\}$ *with the uniform measure.*

We sketch a proof of this statement. Let $\Omega_i = \{\omega_{i1}, \omega_{i2}, \dots, \omega_{im_i}\}$ and let $\mu = \mu_1 \times \dots \times \mu_n$ be the product measure on Ω.

The main idea of the van den Berg-Fiebig theorem is to approximate the probability measures μ_i with sequences of probability measures $(\overline{\mu}_i^n)_{n\geq 1}$ such that

$$k_{ij} := 2^n \overline{\mu}_i(\omega_{ij}) \in \mathbb{N} \cup \{0\}$$

for all $i \in [n]$ and all $j \in [m_i]$. Now the sequence $(\overline{\mu}^n)_{n\geq 1} = (\prod \overline{\mu}_i^n)_{n\geq 1}$ is composed of probability measures on Ω, and as the sets Ω_i are all finite, so that 2^Ω is finite, and as the dyadic rationals $\{\frac{2j+1}{2^k} : j,k \in \mathbb{Z}\}$ are dense in $\mathbb{R}$, we can pick the sequence $\overline{\mu}^n$ so that $\overline{\mu}^n(A) \to \mu(A)$ for all $A \subset \Omega$ (this is called weak convergence of measure).

The proof that, once we have such a sequence of measures, we have proved BK for arbitrary measures on Ω is somewhat tricky and technical, requiring us to essentially mirror the sequence onto a collection of uniform spaces.

4 Butterflies

The discrete hypercube of dimension n is simply the set $Q_n = \{0,1\}^n$. If $x, y \in Q_n$, then the hypercube spanned by x and y is the set $[x,y] = \{z \in Q_n : z_i \in \{x_i, y_i\} \forall i \in [n]\}$. For $z \in Q_n$, we define z^c by $z_i^c := 1 - z_i$. Note that $Q_n = [x, y]$ if and only if $x = y^c$.

The main construction which Reimer uses to prove BK is the idea of a butterfly.

Definition 1. *Given an ordered pair $(a, b) \in (Q_n)^2$, the butterfly on (a, b) is given by the four subcubes*

$$\mathit{Red}(B_{a,b}) := [a, b],\ \mathit{Yellow}(B_{a,b}) := [a, b^c]$$

$$\mathit{Body}(B_{a,b}) := \{a\},\ \mathit{Tip}(B_{a,b}) = \{b\}.$$

The central theorem of Reimer's paper:

Theorem 4.1. *For any flock of butterflies $\mathcal{B}$ with distinct bodies,*

$$|\mathcal{B}| \leq |\mathit{Red}(\mathcal{B}) \cap \mathit{Yellow}^c(\mathcal{B})|.$$

This is the form from which Reimer derives the BK inequality.

As Reimer notes, this inequality is equivalent to saying that the number of Red-Yellow antipodal pairs (i.e. $(r, y) \in \text{Red}(\mathcal{B}) \times \text{Yellow}(\mathcal{B})$ satisfying $[r, y] = Q_n$) is at least the size of the flock. If $\mathcal{B}$ is a flock, then so is $\mathcal{B}' := \{B_{b,a} : B_{a,b} \in \mathcal{B}\}$, and it is not too difficult to see that $\text{Red}(\mathcal{B}') = \text{Red}(\mathcal{B})$ and $\text{Yellow}(\mathcal{B}') = \text{Yellow}^c(\mathcal{B})$. Thus we can prove the Butterfly theorem by proving the form $\mathcal{B} \leq |\text{Red}(\mathcal{B}) \cap \text{Yellow}(\mathcal{B})|$ for any flock $\mathcal{B}$ of butterflies with distinct tips (the passage from $\mathcal{B}$ to $\mathcal{B}'$ turns bodies into tips).

5 Proof of the Butterfly Theorem

Reimer proves the alternate form of Theorem 4.1. through a clever linear-algebraic argument. Given the flock $\mathcal{B}$ of butterflies with distinct tips, let $\overline{Y} := \{x \in Q_n : x \notin \text{Yellow}(\mathcal{B})\}$ and $\overline{R} := \{x \in Q_n : x \notin \text{Red}(\mathcal{B}) \cup \overline{Y}\}$. Then we have the disjoint union

$$Q_n = (\text{Red}(\mathcal{B}) \cap \text{Yellow}(\mathcal{B})) \cup \overline{Y} \cup \overline{R}.$$

Thus $2^n = |\text{Red}(\mathcal{B}) \cap \text{Yellow}(\mathcal{B})| + |\overline{Y}| + |\overline{R}|$. Reimer produces an mapping ψ from the (necessarily disjoint) union $\mathcal{B} \cup \overline{Y} \cup \overline{R} \to \mathbb{R}^{2^n}$ with the property that the image set is linearly independent. As the dimension of $\mathbb{R}^{2^n}$ is 2^n, this implies that the domain of ψ has size at most 2^n. But then the previous decomposition gives us that $|\mathcal{B}| \leq |\text{Red}(\mathcal{B}) \cap \text{Yellow}(\mathcal{B})|$.

Constructing the map is somewhat tricky. We send an element x of $\overline{R}$ to the tensor product $\otimes_{i=1}^{n} e_{x_i}$, where $e_0 = (1,1)$ and $e_1 = (0,1)$. Similar rules are provided for determining the image of $x \in \overline{Y} \cup \mathcal{B}$.

In order to show that the image set is linearly independent, Reimer proves the following six statements:

1. $\psi(\overline{R}) \perp \psi(\overline{Y})$
2. $\psi(\overline{R}) \perp \psi(\mathcal{B})$
3. $\psi(\overline{Y}) \perp \psi(\mathcal{B})$
4. $\psi(\overline{R})$ is independent
5. $\psi(\overline{Y})$ is independent
6. $\psi(\mathcal{B})$ is independent

The difficulty in proving parts 1-5 is mitigated by Reimer's clever construction. To prove part 6, we note that the tips of the butterflies in $\mathcal{B}$ are all distinct, so that for each $y \in Q_n$ there is at most one $B_{x,y} \in \mathcal{B}$ and so at most one $g_{x,y} = \psi(B_{x,y})$ to consider. Thus we restrict ourselves to a set of vectors $\{g_{x(y),y}\}_{y \in Q_n}$ for some arbitrary selection $x : Q_n \to Q_n$. We carry out the computations over $\mathbb{Z}_2$, i.e. we send -1 to 1. This should be more difficult, in principle, than proving independence over $\mathbb{R}$, so if we prove independence over $\mathbb{Z}_2$ we are finished. If we make a matrix with the vectors $g_{x(y),y}$ as columns (so a $2^n \times 2^n$ matrix), after a little manipulation we see that it has a unique column of each possible length. Thus with rearrangement it is an upper-triangular matrix with 1s on the diagonal, hence invertible. Thus the set $\{g_{x(y),y}\}_{y \in Q_n}$ is linearly independent.

This proves the Butterfly Theorem

6 The BK Inequality

Reimer proves the BK inequality using the Butterfly Theorem by means of the Van den Berg-Fiebig theorem.

Theorem 6.1. *Let R and Y be subsets of $\{0,1\}^n$; then $|R||Y| \geq |R \square Y| 2^n$.*

Reimer proves this inequality by taking the flock of butterflies $B_{a,b} \in \mathcal{B}$ and dicing it up according to which subcubes $Q \subset Q_n$ the body $\mathrm{Body}(B_{a,b})$ lies in.

Specifically, he shows that we only need to prove that $|\mathrm{Red}(\mathcal{B})||\mathrm{Yellow}(\mathcal{B})| \geq 2^n |\mathrm{Body}(\mathcal{B})|$, then proves this inequality.

7 Modifications

Borg, Chayes, and Randall change Reimer's proof in many details. They omit all mention of Butterflies or of a Butterfly theorem. They also define the map ψ in a more elegant fashion, sidestepping the somewhat unpleasant piecewise definition which Reimer gives to the reader. Their matrix manipulations to show that the image of ψ is a linearly independent set are more carefully spelled out but also more complicated.

References

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[3] C. Borgs, J. T. Chayes, and D. Randall (1996), The Van den Berg-Kesten-Reimer Inequality: A Review, *Microsoft Research TechReport*, MSR-TR-98-42

[4] D. Reimer (2000), Proof of the Van den Berg-Kesten conjecture, *Combin. Probab. Comput.* **9**, 27–32.