Sharp Thresholds

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March 15, 2010

Abstract

The Kolmogorov Zero-One law states that for tail events on infinite-dimensional probability spaces, the probability must be either zero or one. Behavior analogous in a natural sense to this exists on finite-dimensional spaces as well. Events exhibiting this behavior are said to have a sharp threshold.

Let $Q^n = \{0,1\}^n$ be the discrete hypercube with the probability measure $\mathbb{P}_p$ defined by $\mathbb{P}_p(\omega_1, \omega_2, \ldots, \omega_n) = p^k(1-p)^{n-k}$ where $k = \omega_1 + \omega_2 + \cdots + \omega_n$. An up event $A \subset Q^n$ is symmetric if there exists a transitive permutation group on the indices under which A is invariant. We present the following result, due to Friedgut and Kalai:

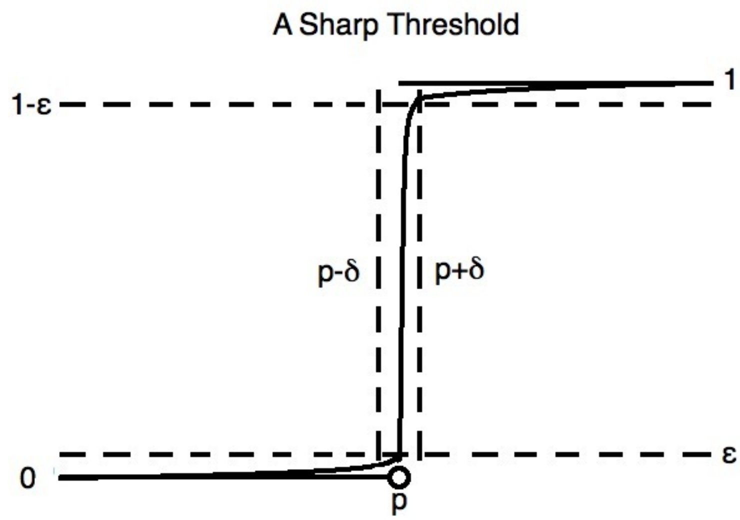
Theorem 0.1. *For every symmetric up-event $A \subset Q^n$, if $\mathbb{P}_p(A) > \epsilon$, then $\mathbb{P}_q(A) > 1-\epsilon$ for $q = p + c\log(1/2\epsilon)/\log n$, where c is an absolute constant (it does not depend on any of the other terms).*

1 Introduction

When the theory of random graphs was first developed by Erdös and Rényi, they noticed that many graph properties exhibited a peculiar property. Their first such encounter was shown in [3], where they demonstrated that for the random graph $G(n,p)$ (the graph of n vertices where each is connected i.i.d. with probability p), if $p < (1-\epsilon)\log n/n$ the graph will almost surely contain an isolated vertex and if $p > (1+\epsilon)\log n/n$, the entire graph will almost surely be connected. Many other graph properties on random graphs were found to exhibit such behavior, which we refer to as a sharp threshold.

In Figure 1, we see an example of a sharp threshold. Here, the thick lines 0 and 1 correspond to the probability of a tail event with critical probability p. The dotted lines show the continuous density function of the probability with respect to p for some given n. Here, δ relies on p, n and ϵ. In general, we can send δ to zero as n goes off to infinity, regardless of the values of ϵ and p.

Figure 1.



Since these first observations, many people have further studied sharp thresholds. They have been demonstrated for a multitude of events. Over the years, the bounds which determine how sharp they are have also improved. In 1996, Friedgut and Kalai demonstrated that every symmetric up-event has a sharp threshold [6] and provided optimal bounds up to a constant factor. Although not the final word in the field, the repercussions of this result are strong.

Applications of sharp thresholds abound. For many problems relating to random graphs, Boolean algebras and other fields, sharp thresholds serve a valuable purpose. If the problem can be shown to be easy when a related event is almost always or almost never true, then we can confine our attention to the narrow band around some particular value of p. Of course, for our purposes the most important application of sharp thresholds relates to percolation. They are at the center of one of the easier approaches to proving Kesten's theorem, which states that $P_H \leq 1/2$. A proof using the result stated below can be found in [2]. We now start developing the tools necessary to prove the sharp threshold theorem of Friedgut and Kalai.

2 Preliminaries

Let A be an event in the hypercube Q^n. For $\omega \in Q^n$, the ith variable ω_i is pivotal if precisely one of $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\overline{\omega} = (\omega_1, \omega_2, \ldots, 1 - \omega_i, \ldots, \omega_n)$ is in A. Note that whether the ith coordinate is pivotal depends both on the point ω and the event A. The influence of the

i*th* variable on A is

$$\beta_i(A) = \mathbb{P}_p(\{\omega \in Q^n : \omega_i \text{ is pivotal for } A\}).$$

The following lemma, first proved by Margulis in 1974 and rediscovered in 1981 by Russo, states the derivative of $\mathbb{P}_p(A)$ with respect to the influence. Recall $A \subset Q^n$ is an up-event if for $\omega = (\omega_1, \ldots, \omega_n) \in A$ and $\alpha_i \geq \omega_i$ for all $i \in [n]$, we have $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$ as well.

Lemma 2.1. *Let $A \subset Q^n$ be an up-event. Then*

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{i=1}^{n} \beta_i(A).$$

This result follows from the fact that as A is an up-event, we can show that for each index $\mathbb{P}_p$ is a constant unless the index is pivotal, in which case it takes the value $p\beta_i(A)$. A proof can be found in [2] p. 46-47.

The next result we need is due to Kahn, Kalai and Linial [7].

Theorem 2.2. *Let $A \subset Q_p^n$ with $\mathbb{P}_p(A) = t$. Then there exists an absolute constant c such that*

$$\max_i \beta_i(A) \geq ct(1-t)\log n/n.$$

When applying Theorem 2.2, we will pull the larger of t and $(1-t)$ inside of our constant as it will be greater than or equal to 1/2. In this way, we only need to concern ourselves with the smaller term. No proof shall be presented, as it relies on finite-dimensional Fourier analysis. While the techniques used are definitional with the exception of Parseval's identity and several inequalities due to Beckner (cited in [7]), it is too involved to present here. A strictly combinatorial proof, also too intricate to be shown here, was presented by Falik and Samorodnitsky in 2005 [4].

We note that both of the above results can be extended. The proof of Lemma 2.1 allows for each index to have its own value p_i. The extension of Theorem 2.2 in 1992 by Bourgain, Kahn, Kalai, Katznelson and Linial [1] is more substantive. Instead of being proved on Q^n, BKKKL states that above holds for an arbitrary n-dimensional probability space. A greatly simplified proof was presented in 2004 by Friedgut [5].

Let $A \subset Q^n$ be an up-event. We say A is symmetric if there is a transitive permutation group Γ on $[n]$ such that A is invariant under Γ. In the case that the event A is symmetric, the influence of each index will be identical. We then denote the influence $\beta(A)$. We are now prepared to prove the sharp threshold theorem of Friedgut and Kalai for symmetric up-events.

3 Result

The following proof is as presented in [6].

Theorem 3.1. *For every symmetric up-event $A \subset Q^n$ with $0 < \epsilon < 1/2$ and $\mathbb{P}_p(A) < \epsilon$, we have $\mathbb{P}_q(A) > 1-\epsilon$ for $q \geq p + c\frac{\log(1/(2\epsilon))}{\log n}$ where c is an absolute constant.*

Proof. Let A be a symmetric up-event. Then the influence of any two indices is the same, with each index having influence $\beta(A)$. Using Lemma 2.1 and Theorem 2.2 with c_1 as in the latter, we compute the following lower bound

$$\frac{d}{dr}\mathbb{P}_r(A) = n\beta(A) \geq nc_1\mathbb{P}_r(A)\frac{\log n}{n} = c_1\mathbb{P}_r(A)\log n.$$

Therefore,

$$\frac{d}{dr}\log\mathbb{P}_r(A) = \frac{\frac{d}{dr}\mathbb{P}_r(A)}{\mathbb{P}_r(A)} \geq \frac{c_1\mathbb{P}_r(A)\log n}{\mathbb{P}_r(A)} = c_1\log n. \tag{1}$$

For p such that $\mathbb{P}_p(A) \geq \epsilon$, define $q' = p + \frac{\log(1/(2\epsilon))}{c_1\log n}$. Then

$$\log(\mathbb{P}_{q'}(A)) \geq \log(\mathbb{P}_p(A)) + \int_p^{q'} c_1\log n dr \geq \log(\epsilon) + \log(1/(2\epsilon)) = \log(1/2).$$

We are now half of the way there. The rest of the way uses the same approach, with one slight variation.

For $1/2 \leq \mathbb{P}_r(A) < 1-\epsilon$, our lower bound changes as follows:

$$\frac{d}{dr}\mathbb{P}_r(A) = n\beta(A) \geq nc_1(1-\mathbb{P}_r(A))\frac{\log n}{n} = c_1(1-\mathbb{P}_r(A))\log n.$$

Then

$$\frac{d}{dr}\log(1-\mathbb{P}_r(A)) = \frac{\frac{d}{dr}(1-\mathbb{P}_r(A))}{(1-\mathbb{P}_r(A))} \leq -\frac{c_1(1-\mathbb{P}_r(A))\log n}{(1-\mathbb{P}_r(A))} = -c\log n.$$

Note the direction of the inequality changes compared to Equation 1 as we are now taking the derivative of $-\mathbb{P}_r(A)$. Then by defining $q = q' + \frac{\log(1/(2\epsilon))}{c_1\log n}$, we see

$$\log(1-\mathbb{P}_q(A)) \leq \log(1-\mathbb{P}_{q'})(A)) - \int_{q'}^{q} c_1\log n\ dr \leq \log(1/2) - \log(1/(2\epsilon)) = \log(\epsilon).$$

Thus for $c = 2c_1$ and $q = p + \frac{\log(1/(2\epsilon))}{c\log n}$ we have $\mathbb{P}_q(A) > 1-\epsilon$. $\square$

The above result can be generalized in several ways. When p does not rely n, the above bound is sharp except for improvements upon the constant c. If p decreases with n, then the bound can be improved by using the value $q = p + cp\log(1/p)\frac{\log(1/(2\epsilon))}{\log n}$. It can also be adapted to probabilities with a finite number of possible values, so long as all but one of these values has appropriately small probability. Further generalizations can be found in [6].

References

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