Renormalization: An Attack on Critical Exponents

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1   Introduction

Suppose $L \subset \mathbb{R}^d$ is a lattice with critical probability p_c. Percolation on L is significantly differs depending on $p < p_c$ or $p > p_c$. The following table details some of the striking differences between the subcritical and supercritical percolation.

Quantity	$p < p_c$	$p > p_c$
Number of infinite clusters	0	1
Percolation probability	$\theta(p) = 0$	$\theta(p) > 0$
Expected cluster size	$\chi(p) < \infty$	$\chi(p) = \infty$
Tail of finite clusters	$\mathbb{P}_p(n \leq |C| < \infty) \approx e^{-an}$	$\mathbb{P}_p(n \leq |C| < \infty) \approx e^{-bn^{(d-1)/d}}$

These results suggest the possibility of unexpected phenomena when p is very close to p_c. Perhaps these discrepancies will interact in an interesting manner when p is close to p_c. In other words, if $f(p)$ is some quantity which exhibits differing behavior on whether $p < p_c$ or $p > p_c$, we seek to understand $f(p)$ as $p \to p_c$.

Despite the simplicity of the task at hand, mathematicians have made very little headway in understanding the percolation process at or near p_c. Although theoretical physicists have made grandiose predictions and heuristic arguments (backed by much data), very few results are actually mathematically rigorous. The two essential techniques used for understanding critical phenomena, *scaling theory* and *renormalization*, were initially pioneered by applied scientists and their mathematical underpinnings remain uncertain. These techniques give arguments which demonstrate a number of the crowd-pleasing conjectures (we will get to these shortly), though they all rely on an unproven underlying assumption. Regarding this, Grimmett [3] writes "The challenge to mathematicians is therefore to make sense of scaling theory, rather than to verify a bunch of conjectures."

We now describe the various quantities which exhibit interesting behavior for p close to p_c. In what follows, we focus specifically on the lattice $L = \mathbb{Z}^d$ for simplicity, however, everything can easily be generalized to an arbitrary lattice in d-dimensions. The remarkable series of conjectures is that all of the quantities we shall mention satisfy certain types of *power* growth near p_c. They key piece of information to take away from such a quantity will be its *critical exponent* (these will be described in detail). Even more stunning is the belief that these critical exponents do not actually depend on the underlying lattice. Physicists believe they should depend only on the dimension of the ambient space. If true, then in some sense the critical exponents hold more power than the critical probability p_c, as there are many examples of lattices of the same dimension with distinct critical probabilities.

We know that the percolation probability, $\theta(p)$, is positive for $p > p_c$. It is conjectured that $\theta(p)$ exhibits a power-like behavior for p near p_c. Specifically, it is believed that there exists $\beta > 0$ so that

$$\theta(p) \approx (p - p_c)^\beta \quad \text{as } p \downarrow p_c.$$

Before continuing, we should explain what we mean by the symbol "$\approx$". The standard asymptotic statement

$$\theta(p - p_c)^{-\beta} \to 1 \quad \text{as } p \downarrow p_c$$

is actually too strong. We only wish to say that $\theta(p)(p - p_c)^{-\beta}$ is bounded away from 0 and infinity as $p \downarrow p_c$. In other words, $\theta(p)$ behaves like constant times $(p - p_c)^\beta$ close to p_c. Another way of expressing this conjecture is that

$$\lim_{p \downarrow p_c} \frac{\log \theta(p)}{\log(p - p_c)} = \beta,$$

and henceforth this is what we shall mean by $\approx$. Note this conjecture actually implies percolation does not occur at p_c. This is known to be the case only when $d = 2$ and $d \geq 19$, and the question remains open for all other d.[1]

The expected cluster size is also conjectured to satisfy a power growth near p_c. Specifically, it is conjectured that there exists $\gamma > 0$ so that

$$\chi(p) \approx (p_c - p)^{-\gamma} \quad \text{as } p \uparrow p_c.$$

In addition to $\theta(p)$ and $\chi(p)$, there are several other quantities which are conjectured to exhibit power behavior near p_c. Well known examples are the expected size of the finite cluster (which equals $\chi(p)$ when $p < p_c$) and the expected number of open clusters per vertex.

There are also several functions which are conjectured to satisfy power growth at $p = p_c$. Assuming $\theta(p_c) = 0$, we know that

$$\infty = \chi(p_c) = \sum_{n=1}^{\infty} n \mathbb{P}_{p_c}(|C| = n).$$

Therefore the quantity $\mathbb{P}_{p_c}(|C| = n)$ cannot decay exponentially, for otherwise the above sum would converge. This suggests $\mathbb{P}_{p_c}(|C| = n)$ behaves like a negative power of n. It is conjectured that there exists $\delta \geq 1$ so that

$$\mathbb{P}_{p_c}(|C| = n) \approx n^{1-1/\delta}.$$

Likewise, the probability that there exists an open path from the origin to the surface of a box of size n as well as the probability that a given cluster has radius n are both conjectured to decay like a power of n at $p = p_c$.

We give an important example of a quantity which depends on both n and p. Let E_n be the event that "the origin and the point $(n, 0, ..., 0)$ are in the same finite open cluster". Let $\tau_p(n) = \mathbb{P}_p(E_n)$. It is conjectured that there exists some constant η and some function $\xi(p)$ so that

$$\tau_p(n) = \begin{cases} n^{2-d-\eta}, & \text{if } p = p_c \\ e^{-n/\xi(p)}, & \text{if } p \neq 0, 1, p_c, \end{cases}$$

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[1] Many near-critical percolation results are actually rigorous for large d. The idea is that when d is large enough, percolation on the lattice in question is similar to percolation on a binary tree. This idea has been made rigorous for $d \geq 19$, although physicists believe the technique should work for $d \geq 7$.

where $\xi(p)$ satisfies $\xi(p) \to \infty$ as $p \to p_c$. For us, $\xi(p)$ is an important quantity known as the *correlation length*. This conjecture agrees with the general belief that any reasonable percolation quantity should behave like a power at or near the critical point. Consider $\tau_p(n)$ in the double limit $p \to p_c$ and $n \to \infty$. For a fixed probability p, we know that $\tau_p(n) \to 0$ as $n \to \infty$. On the other hand, for fixed n, we actually know that $\tau_p(n)$ is monotonically increasing as $p \to p_c$. Thus, one might expect interesting behavior in the double limit. This discussion is somewhat analogous to the limit

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

The base $1 + 1/n$ is a real number strictly greater than 1, so we might expect that this limit tends to infinity since we are raising it to higher powers. Conversely, if we raise a number very close to 1 to the positive power n, then result remains close to one. This seems to suggest the limit is 1. Instead, a compromise is reached and $(1 + 1/n)^n \to e$.

In our case, we might guess that

$$\frac{\tau_p(n)}{\tau_{p_c}(n)} \approx e^{n/\xi(p)} \quad \text{for } p \text{ near } p_c \text{ and large } n.$$

That is to say, $\tau_p(n)$ and $\tau_{p_c}(n)$ differ significantly only when $n/\xi(p)$ is large. So $\xi(p)$ is the natural length scale of bond percolation. The function $\xi(p)$ is the minimal scale on which edge percolation at p differs from edge percolation at p_c. Again, it is conjectured that there exists a constant $\nu > 0$ so that

$$\xi(p) \approx |p - p_c|^\nu \quad \text{as } p \to p_c.$$

These critical exponents β, γ, and ν seem unrelated. However, results from scaling theory imply that these quantities actual satisfy linear and quadratic equations known as the *scaling* and *hyperscaling* relations. Moreover, physicists conjecture $\beta = 5/36$, $\gamma = 43/18$, and $\nu = 4/3$. In the case when L is the triangular lattice in the plane, Smirnov's theorem rigorously proves these results [2, 4].

2 Renormalization

Renormalization is a general method used in a variety of settings by theoretical physicists and applied mathematicians. Usually a renormalization procedure rests on several assumptions which cannot be made rigorous. However, there are situations in which renormalization has solid foundations. For example, through the guise of k-dependence, such techniques are rigorously used in the proof of Kesten's theorem [2]. We use renormalization as a method for understanding critical exponents. In particular, renormalization demonstrates (though not rigorously) several of the scaling relations among critical exponents. In addition, it allows one to approximate both the critical exponents and the critical probability p_c computationally.

We restrict ourselves to the lattice $L = \mathbb{Z}^2$, though everything that follows extends almost verbatim for $L = \mathbb{Z}^d$, and without much additional trouble for arbitrary lattices. Our first task will be to partition $\mathbb{Z}^2$ into blocks of size b, where b is some positive integer. Specifically, the blocks in this partition will be sets of the form

$$B(x, y) = \left\{(u, v) \in \mathbb{Z}^2 \,\middle|\, xb \le u < (x+1)b, \, yb \le v < (y+1)b\right\}$$

where $(x, y) \in \mathbb{Z}^2$. We give create a new lattice from these blocks, where each block is now a vertex and two blocks are adjacent if there exists vertices in $\mathbb{Z}^2$ within each block that are adjacent in $\mathbb{Z}^2$. We call this new lattice $\mathcal{R}_b$ the *renormalization of* $\mathbb{Z}^2$. We must now describe how the percolation process on $\mathbb{Z}^2$ induces a percolation on $\mathcal{R}_b$. We say that a block $B(x_0, y_0)$ is *traversable in the horizontal direction* if there exists an open path contained entirely with $B(x_0, y_0)$ from some vertex of the form (bx_0, y) to some vertex of the form $((b+1)x_0 - 1, y')$ with the edge between $((b+1)x_0 - 1, y')$ and $((b+1)x_0, y')$ open. Put another way, there must exist an open path within $B(x_0, y_0)$ from some vertex on the left side of the block to some vertex on the right side of the block, and an additional edge leading out to the right. A similar definition describes when a block is traversable in the vertical direction. In the lattice $\mathcal{R}_b$, we say that a horizontal (resp. vertical) edge is open if the left-hand (resp. bottom) block of the two blocks incident to the edge is traversable horizontally (resp. vertically).

We now have a percolation process on $\mathcal{R}_b$ in which each edge is open with probability $R_b(p) := \mathbb{P}_p(B(0,0)$ is traversable in the horizontal direction). How does $R_b(p)$ change with p? For small enough b, it is not difficult to compute $R_b(p)$ by hand. For example, when $b = 2$ we know that the event that a block is traversable horizontally is only dependent on six edges in $\mathbb{Z}^2$. In the Figure 1, $R_2(p)$ is the probability that there exists an open path from a vertex on the left side of the graph to a vertex on the right side of the graph. Since there are only $2^6 = 64$ possible configurations of open and closed edges, we could make a list of all possible configurations and determine all cases in which the block is horizontally traversable. However, we see that this soon becomes unwieldy as b grows larger.

Figure 1: The edges on which the event "$B(0,0)$ is traversable horizontally" is dependent.
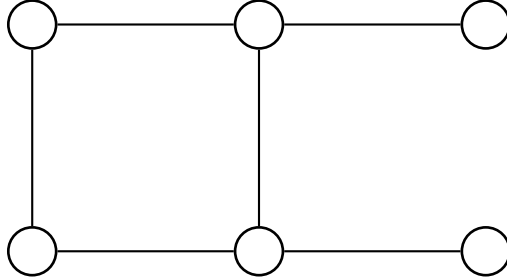
What can we say regarding $R_b(p)$ for large b? If p is close to 0 and the block size b is large, we expect $R_b(p)$ to be much smaller than p, since we are requiring at least b open edges in a block of size b^2, and moreover, these open edges must be aligned in just the right way to guarantee an open path across the block. Similarly, if p is close to 1, then by duality the same argument shows $R_b(p)$ should be larger than p. Thus we expect the graph of $R_b(p)$ to be some sort of S-shape curve with a unique nontrivial fixed point $\tilde{p}$. See Figure 2.

Note that in the original lattice $\mathbb{Z}^2$, bond percolation is independent (the event that one edge is open is independent from the event that a second edge is open). This is no longer true in $\mathcal{R}_b$. For example, knowing that a block is traversable in the horizontal direction increases the likelihood that it is also traversable vertically, since the former implies at least b open edges within the block. Despite this, renormalization assumes two hypotheses regarding the renormalized lattice $\mathcal{R}_b$:

1. Bond percolation on $\mathcal{R}_b$ closely mimics bond percolation on $\mathbb{Z}^2$ with edge probability $R_b(p)$
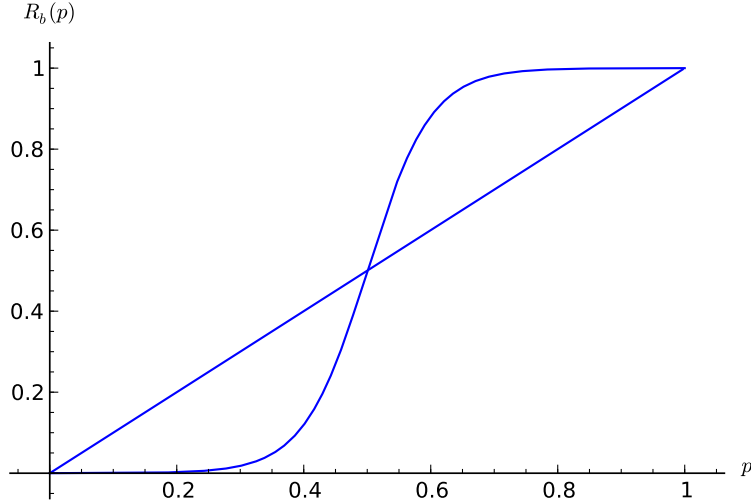
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Figure 2: This shows the expected S-shape curve related p with $R_b(p)$. Note the unique nontrivial fixed point $\tilde{p}$ and that $R_b'(\tilde{p}) > 1$.

    2. The large-scale connectivities of these two processes are similar

These are two fundamental assumptions of renormalization which, though plausable and intuitive, remain unproven.

    As previously discussed, $\xi(p)$ is the minimal scale over which bond percolation at p can be distinguished from bond percolation at p_c. By our first hypothesis, we know that the correlation length on the renormalized lattice $\mathcal{R}_b$ is $\xi(R_b(p))$. Because each unit of length in $\mathcal{R}_b$ is actually b units of length in $\mathbb{Z}^2$, the second hypothesis implies the equation

$$\xi(p) = b\xi(R_b(p)). \tag{1}$$

Evaluating (1) at the fixed point $\tilde{p}$, we have $\xi(\tilde{p}) = b\xi(\tilde{p})$. Because b is a finite number, it must be the case that $\xi(\tilde{p}) = 0, \infty$. Because it is not the case that $\xi(\tilde{p}) = 0$, we have $\xi(\tilde{p}) = \infty$. But p_c is the unique value of p for which $\xi(p) = \infty$. Thus this fixed point $\tilde{p}$ is actually the critical probability for the lattice $\mathbb{Z}^2$.

    One could use the above heuristic as a recipe for estimating p_c. Because the two renormalization hypotheses are imprecise and should only be regarded an approximation of the truth, we cannot expect that $\tilde{p}$ exactly equals p_c. However, for larger and larger values of b, it is plausible that $\tilde{p}$ approaches p_c. The computational difficulty here is the aforementioned problem of computing $R_b(p)$ for large b.

3   Obtaining a critical exponent

Using equation (1) and that $\tilde{p} = p_c$ as derived from the renormalization hypotheses, we can obtain information regarding critical exponents. Again, our derivations here have somewhat of a "hand-wavy" feel. This should not trouble the reader since our assumptions are already based on dubious

hypotheses. Let $\lambda = R'_b(p_c) > 1$ as previously mentioned (see Figure 2). We linearize $R_b(p)$ at $p = p_c$ to obtain

$$p - p_c \simeq \lambda(p - p_c) \quad \text{as } p \to p_c. \tag{2}$$

Let $\psi(p - p_c) = \xi(p)$ for simplicity. Substituting (2) in for (1), we have

$$\psi(p - p_c) \simeq b\psi(\lambda(p - p_c)) \quad \text{as } p \to p_c. \tag{3}$$

Thus equation (3) gives a recurrence for ψ and hence ξ. Iterating (3) m times, we have

$$\psi(p - p_c) \simeq b^m \psi(\lambda^m(p - p_c)) \quad \text{as } p \to p_c.$$

Now choose a positive constant A. Because $\lambda > 1$, there exists m so that $\gamma^m|p - p_c| \simeq A$. After several algebraic manipulations (see [3]), we find that $\psi(p - p_c) \approx D|p - p_c|^\nu$ where D is a constant depending only on b, λ, and A, and

$$\nu = \frac{\log b}{\log \lambda}. \tag{4}$$

This last statement is the punchline we have sought. It shows that indeed $\xi(p)$ satisfies power growth near p_c.

In analogy with our derivation that $\tilde{p} = p_c$, there are two possible directions in which to turn given (4). Computation of the function $R_b(p)$ allows one to estimate ν. More "theoretically", one may derive similar expressions for other critical exponents using the same renormalization framework. It is these expressions that give rise to the scaling and hyperscaling relations.

References

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