Two Proofs of Commute Time being Proportional to Effective Resistance

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Abstract

Chandra et al. have shown that the effective resistance, R_{uv} between two nodes, u and v of an electrical network is directly proportional to the commute time, C_{uv} of these two vertices on an equivalent undirected graph with edges being replaced by resistors. The constant of proportionality varies depending on the resistance and cost of each edge. Two proofs are given to show that $C_{uv} = 2m.R_{uv}$ where m is the total number of edges.

1 Introduction

Doyle and Snell [1] showed that a random walk on an undirected graph, G with vertices, V and edges, E are related to voltages in an electrical network. This relation becomes clear if each edge is replaced by a resistor with resistance, $r_{xy} > 0$. It can, then be shown that the probability of reaching y from vertex x before reaching some vertex z, P_x is the voltage at x with respect to ground. This was proved by showing that both P_x and voltage at x with respect to ground are harmonic functions satisfying the same boundary conditions. As a reminder, a random walk would start at a vertex and would choose where to go next by selecting a vertex uniformly at random.

The hitting time, H_{uv} is defined to be the expected number of steps a random walk would take starting at vertex u and first arriving at v. Commute time, C_{uv} is then defined to be $C_{uv} = H_{uv} + H_{vu}$ which is the expected number of steps for a random walk starting at u to hit v and to hit u back. Section 4 would show in detail the proofs given in [2] about how C_{uv} is related to the effective resistance R_{uv}.

In general, the effective resistance, R_{xy} is defined to be the voltage measured between x and y with a unit current sent into node x and one unit of current removed from node y.

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2 Goal of this Paper

The aim is to provide two proofs from [2] of Theorem 2.1.

Theorem 2.1. *For any two vertices, u and v in an undirected connected graph, $G = \{V, E\}$ with $|V| = n$ vertices and $|E| = m$ edges, where each edge has unit resistance and unit cost, then $C_{uv} = 2mR_{uv}$.*

3 Preliminaries

The voltage across a branch xy is denoted by ψ_{xy}. I_{xy} is the current flowing from x to y. If this current is positive the x terminal would be more positive than the y terminal. The three laws that we need for the first proof are:

- Ohm's Law states that the voltage across a resistor is proportional to the current across it.
- Kirchhoff's Voltage Law states that the voltage around a closed loop is zero. For example, consider the loop, L, in Figure 1. The following is true: $\psi_{DC} + \psi_{CE} + \psi_{FE} + \psi_{FD} = 0$.
- Kirchhoff's Current Law states that the current entering a node is equal to the current going out. For example, in Figure 1, at node C, $I_{DC} + I_{AC} = I_{CE}$

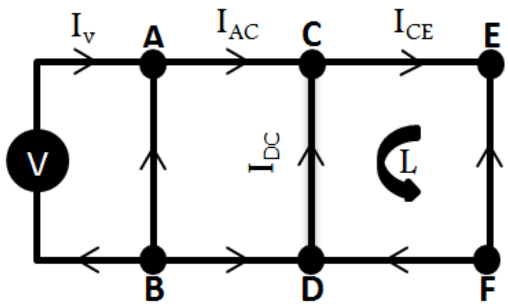

Figure 1: Illustrating Kirchhoff's Voltage and Current Laws

Assume that G is a connected graph with each edge having a resistance and a cost, denoted by r_{xy} and f_{xy} respectively. Conductance of each edge, (x, y), is defined to be $C_{xy} = \dfrac{1}{r_{xy}}$.

After a random walk the probability of transitioning from vertex, x to another neighboring one, y is given by $P_{xy} = \dfrac{C_{xy}}{C_x}$ where $C_x = \sum_y C_{xy}$. Finally, $N(x)$ denotes the neighbors of x.

4 Proofs

The first proof is by Chandra et al. [2] and uses concepts borrowed from electrical network. The second proof is also from [2].

Proof. 1

Input $d(x)$ units of current into each node x. This means there would be $2m$ units of current flowing which would be removed from node v (Kirchhoff's Current Law) (refer to Figure 2).

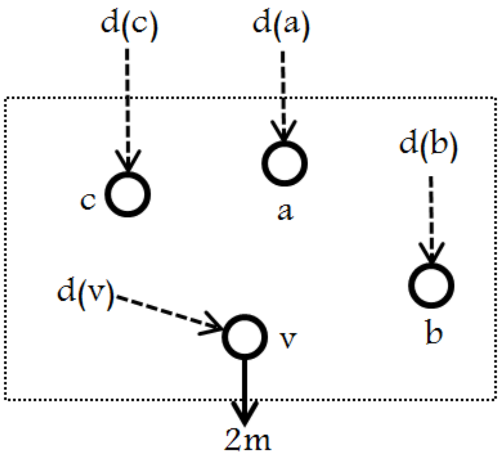

Figure 2: $d(x)$ units of current flowing to each node with $2m$ units removed from node v

Applying Kirchhoff's Voltage Law we get that $\psi_{xy} = \psi_{xv} - \psi_{yv}$ since $\psi_{vv} = 0$. By Ohm's law the current flowing across an edge, $(x, y) \in E$, I_{xy} is given by equation 1.

$$I_{xy} = \frac{\psi_{xv} - \psi_{yv}}{r_{xy}} = \psi_{xv} - \psi_{yv} \tag{1}$$

Current flowing from any vertex, u where $u \neq v$ to vertex x, I_{ux}, can be computed by applying Kirchhoff's Current Law and is shown in equation 2.

$$\sum_{x \in N(x)} I_{ux} = d(u) \tag{2}$$

Combining equations (1) and (2) and with proper substitution of subscripts, equation 3 results.

$$d(u) = \sum_{x \in N(u)} (\psi_{uv} - \psi_{xv}) \tag{3}$$

$$= \sum_{x \in N(u)} \psi_{uv} - \sum_{x \in N(u)} \psi_{xv} \tag{4}$$

$$= d(u).\psi_{uv} - \sum_{x \in N(u)} \psi_{xv} \tag{5}$$

Rearranging equation 5,

$$\psi_{uv} = 1 + \frac{1}{d(u)} \sum_{x \in N(u)} \psi_{xv} \tag{6}$$

But hitting time, H_{uv},

$$H_{uv} = 1 + \frac{1}{d(u)} \sum_{x \in N(u)} H_{xv} \tag{7}$$

And, equation (6) has the same series of linear equations as equation (7) with different parameters. For, equation (8) to be true there must exist a unique solution. This can be proved as follows. Let's assume that there are in fact two solutions, μ_u and $\bar{\mu}_u$ where one would be larger. Now, if we propagate this to its neighbors, then they would also have one value larger than the other, but at $u = v$ we get H_{vv} $= \psi_{vv} = 0$ which is a contradiction. So, equation 8 holds.

$$H_{uv} = \psi_{uv} \tag{8}$$

Now, for convenience, let's fix a vertex, u and instead of removing $2m$ units of current from v remove it from u as shown in Figure 3.

Add the negative of the flows in Figure 3 to those in Figure 2 and, let's denote the resulting flow by F. Then,

$$\forall x, F = \psi_{xv} - \psi_{xu} \tag{9}$$

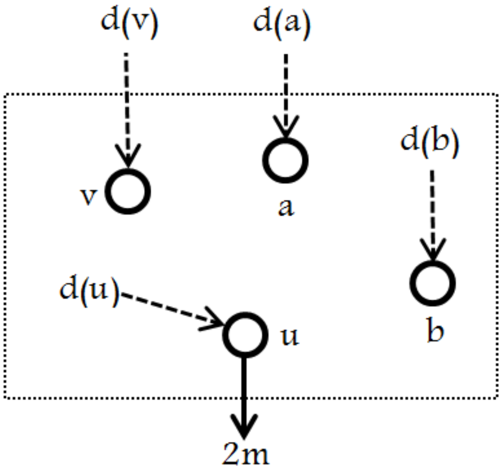

Figure 3: $2m$ units of current flowing out of u instead of node v

Equation 9 shows that there are $2m$ units of current entering u and leaving v. Note that the internal flows cancel each other. From the definition of C_{uv},

$$C_{uv} = H_{uv} + H_{vu} \tag{10}$$

$$= \psi_{uv} + \psi_{vu} \tag{11}$$

$$= (\psi_{uv} - \psi_{uu}) - (\psi_{vv} - \psi_{vu}) \tag{12}$$

$$= \psi_{uv} + \psi_{vu} \tag{13}$$

Equation 12 results from substituting x for u and then x for v in equation 9.

Now, R_{uv} is defined for 1 unit flowing from u to v, and here we have $2m$ units, so by Ohm's Law, $R_{uv} = \dfrac{\psi_{uv} + \psi_{vu}}{2m}$.
And,

$$C_{uv} = 2m.R_{uv} \tag{14}$$

□

Proof. 2
Consider a u–to–u excursion with time T as shown in Figure 4. Then the expected time of hitting any node x, $E[N_x]$ is equal to the expected time of T, $E[T]$ times the stationary probabiblity of being at x and is given by equation 15. This is proven in [3] which explains that each return to node u is a renewal i.e. each time a node u is re-visited it's as if the random walk starts over again. Consequently, it can be proven that the proportion of time at a given node, x is equal to the ratio of the expected time of visiting, x in a u–to–u cycle and the expected time of that cycle.

$$E[N_x] = E[T].\pi_x \tag{15}$$

Since every edge has a unit cost and a unit resistance, we get $\pi_u = \frac{d_u}{2m}$ for $x = u$. Now, we

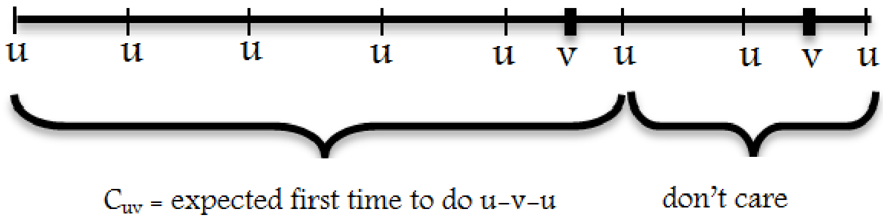

Figure 4: a u−to−u excursion where we are interested in only the first time node v is hit

let T be the time of first return to u after hitting node v. But, by definition this is also C_{uv}. So, equations 16 and 17 follow.

$$E[N_u] = E[T].\frac{d_u}{2m} \tag{16}$$
$$= C_{uv}.\frac{d_u}{2m} \tag{17}$$

$E[N_u]$ is the expected time that we do a u−to−u excursion until node v is hit. Remember that we do return to node u after hitting v. Now, the probability of hitting v before returning to u is just the probability of escape, P_{esc} and by definition is given by equation 18.

$$P_{esc} = \frac{1}{d_u.R_{uv}} \tag{18}$$

Let R be a geometric variable which denotes the number of trials until a success happens in a sequence of independent bernouilli trials. Here, a success is the trial where during a random walk from u, we hit v. This is shown in Figure 5. Failures occur when node v is not hit.

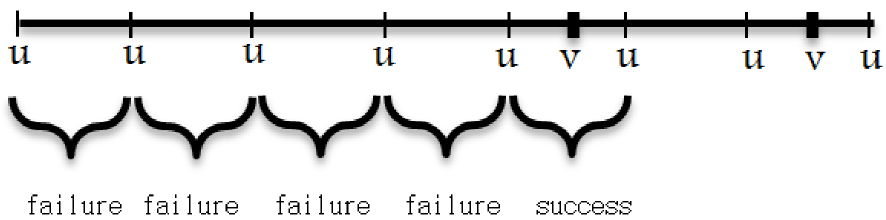

Figure 5: a u−to−u excursion where failures and success are marked

Denoting the probability of success by p then,

$$E[R] = \frac{1}{p} = d_u.R_{uv} \tag{19}$$

And, by definition $E[R] = E[N_u]$.

$$C_{uv}.\frac{d_u}{2m} = d_u.R_{uv} \tag{20}$$

Rearranging equation 20,

$$C_{uv} = 2m.R_{uv} \tag{21}$$

□

5 A Generic Equation

Theorem 2.1 is only valid if each of the edge, $(x, y) \in E$ has resistance, r_{xy} of one ohm and a cost, f_{xy} of one. A general equation can be proven by injecting a current of $\sum_{y \in V} \frac{f_{xy}}{r_{xy}}$into each node x and gives Theorem 5.1 which is taken from [2].

Theorem 5.1. *For any two vertices, $u, v \in G$, $C_{uv} = \sum_{(x,y) \in V \times V} \frac{f_{xy}}{r_{xy}}.R_{uv}$*

Table 1 gives a summary of the commute time for different costs and resistances of each edge.

Table 1: Commute Time -
Varying Cost and Resistance

f_{xy}	r_{xy}	C_{uv}	Definition
1	1	$2m.R_{uv}$	m = number of edges
1	r_{xy}	$C.R_{uv}$	$C = \sum_{(x,y) \in V \times V} C_{xy}$
r_{xy}^2	r_{xy}	$r.R_{uv}$	r = $\sum_{(x,y) \in V \times V} r_{xy}$

6 Conclusion

In this paper two proofs of Chandra's Theorem on commute time have been given and a more generic equation has been stated which has a similar proof to *Proof 1.*

References

[1] Peter G. Doyle, J. Laurie Snell, Random Walks and Electric Networks, version 2006

[2] Ashok K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, P. Tiwari, The Electrical Resistance of a Graph Captures its Commute and Cover Times, 1996

[3] Sheldon M. Ross, Introduction to Probability Models, 9th Edition

[4] John Hopcroft, Mathematical Foundations for Information Age, Lecture 24, 2009