Random Walks and Foster's Resistance Theorem

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Abstract

Given any simple connected graph $G =< V, E >$ such that each edge $e \in E$ has unit resistance, Foster's Resistance Theorem states that the summation of effective resistances over all edges equals $n - 1$. That is,

$$\sum_{e \in E} R_e = n - 1$$

Further, given a general graph with resistances r_e corresponding to edges $e \in E$, the theorem states:

$$\sum_{e \in E} \frac{R_e}{r_e} = n - 1$$

Tetali[1] proved Foster's Resistance Theorem by considering traversals and then appling the Reciprocity theorem. In this paper, we go through that proof.

1 Introduction

Doyle and Snell [1] discovered a connection between electrical networks and random walks in graphs. At the time, the discovery was novel and led to breakthroughs in both avenues of research. Tetali [2] related the effective resistance R_e for $e \in E$ to traversals in a random walk. This idea was inspired by the commute time [3]; and, along with the Reciprocity Theorem and Kirchoff's Current Law, the idea was used to prove Foster's Resistance Theorem.

2 Preliminaries

Consider a *simple network* $G =< V, E >$ where every edge $e \in E$ has the same resistance r_e. For a pair of vertices $v_i, v_j \in V$, we define $P_{ij} = \text{Prob}[v_i \text{ can reach } v_j \text{ in one step}] = 1/d(v_i)$ if $(v_i, v_j) \in E$ and $P_{ij} = 0$ otherwise.

3 Theorems

Theorem 3.1. *In a simple network where each edge has unit resistance, the effective resistance $R_{i,j}$ between any pair of vertices $v_i, v_j \in V$ is equal to the expected number of traversals along any specific edge (v_i, v_k) before arriving at v_j in a random walk from v_i to v_j.*

Proof. Let $U_k^{i,j}$ be the expected number of times we hit k before reaching j on a random walk from i to j. To hit k, we must first hit a neighbor of k and then cross over an edge to k. Therefore,

$$U_k^{ij} = \sum_w U_w^{ij} P_{wk} = \sum_w \frac{U_w^{ij}}{d(w)}$$

since for $w \in N(k)$ we have $P_{wk} = \frac{1}{d(w)}$.

Next, we divide both sides by $d(k)$ to get:

$$\frac{U_k^{ij}}{d(k)} = \frac{1}{d(k)} \sum_w \frac{U_w^{ij}}{d(w)}$$

Let V_{kj} denote the voltage difference between k and j. Then, we have:

$$(\star) \quad V_{kj} = \frac{U_k^{ij}}{d(k)}$$

Applying Kirchoff's Current Law, we get:

$$V_{kj} = \sum_w \frac{1}{d(k)} V_{wj} \qquad where\ v_w \neq v_i \wedge v_w \neq v_j$$

Applying the definition of effective resistance over edge $R_{ij} = \frac{V_{ij}}{r_{ij}}$ where $V_{kw} = V_{kj} - V_{wj}$, we find:

$$i_{kw} = (V_{kj} - V_{wj})/1 = \frac{U_k^{ij}}{d(k)} - \frac{U_w^{ij}}{d(w)}$$

This completes the proof. □

Theorem 3.2. *The effective resistance R_{ij} is equal to the expected number of edge traversals out of v_k along any specific edge $(v_k, v_w) \in E$ where $v_k \neq v_i \wedge v_k \neq v_j$.*

Proof. Let U_k denote the expected number of visits to k in a random walk that starts at v_i, goes to v_j, and returns back to v_i. It follows that the $U_k = U_k^{ij} + U_k^{ji}$ Applying $\star$, we get

$$U_k = U_k^{ij} + U_k^{ji} = V_{kj}d(k) + V_{ik}d(k)$$

$$= V_{kj}d(k) - V_{ki}d(k) = V_{ij}d(k)$$

$$\longrightarrow R_{ij} = \frac{U_k}{d(k)}$$

Recall that $\frac{U_k}{d(k)}$ is the expected number of visits to k along any specific edge (k, z) where $z \in N(k)$ in our random walk. The theorem follows.

□

Given a graph $G =< V, E >$, suppose we consider two electrical setups in which we give special attention to four vertices i, j, k, and w with $(i, j) \in E$ and $(k, w) \in E$. In the first setup, setup 1, we apply a battery across vertices i and j with a voltage difference $V^1_{i,j}$ that admits a unit current from k to w. In the second setup, we apply a battery across vertices k and w with voltage difference $V^2_{k,w}$ that admits a unit current from i to j.

Theorem 3.3. *In any single source connected graph, by the Reciprocity Theorem [4] we have* $V^1_{i,j} = V^2_{k,w}$.

We now consider two setups for the given graph. The first has source v_i, sink v_j, and a voltage is applied so that the current flowing from v_k to v_j is a unit current. In the second setup we have source v_k, sink v_j, and a voltage is applied so that the current flowing from v_i to v_j is a unit current. Let V_{ij} and V_{kj} be these respective voltages. By $\star$ and the previous theorem, we have:

$$V_{ij} = V_{kj} \leftrightharpoons \frac{U_k^{ij}}{d(k)} = \frac{U_i^{kj}}{d(i)}$$

The right equation refers to random walks starting from v_k and v_i respectively and ending at v_j. The expected number of times we reach v_j from one of its neighbours when $v_k \neq v_j$ is exactly 1. Therefore,

$$\sum_{v_k \in N(v_j)} \frac{U_k^{ij}}{d(k)} = 1 \quad v_i \neq v_j$$

We apply the result above to the Reciprocity Theorem to get:

$$\sum_{v_k \in N(v_j)} \frac{U_k^{ij}}{d(k)} = \sum_{v_k \in N(v_j)} \frac{U_i^{kj}}{d(i)} = \frac{d(k)}{d(k)} = 1$$

Since $U_j^{ij} = 0$, we can sum over all vertices to get:

$$\sum_{v_i \in V} \sum_{v_k \in N(v_j)} \frac{U_k^{ij}}{d(k)} = n - 1$$

Rewriting the previous equation, we get:

$$\sum_{(v_i, v_j) \in E} [U_k^{ij}/d(k) + U_i^{kj}/d(i)] = n - 1$$

Applying Theorem 3.2, we complete Tetali's proof of Forester's Resistance Theorem[4]:

$$\sum_{(i,j) \in E} R_{i,j} = n - 1$$

4 General network

In order to extend Foster's Resistance Theorem to general networks, we begin by defining conductance $c_e = 1/r_e$ for edges $e \in E$ and 0 along all other edges. For a vertex x, let $c(x)$ be the sum of the conductances of edges incident on x. Rewriting $\star$, we see:

$$\star\star V_{kj} = \frac{U_k^{ij}}{c(k)}$$

We have:

Theorem 4.1.

$$\sum_{<v_i, v_j> \in E} \frac{R_{ij}}{r_{ij}} = n - 1$$

Proof. The proof starts by applying the Reciprocity Theorem[4]

$$U_k^{ij}/c(k) = U_i^{kj}/c(i)$$

Multiplying both sides by c_{ij} we get

$$\sum_{v_k \in N(v_i)} U_k^{ij} c_{ij}/c(k) = \sum_{v_i \in N(v_j)} U_i^{kj} c_{ij}/c(i) = 1 \quad if k \neq j$$

$$\longrightarrow \sum_{v_j \in G} \sum_{v_i \in N(v_i)} U_k^{ij} c_{ij}/c(k) = n-1$$

$$\longrightarrow \sum_{\{v_i, v_j\} \in G} [U_k^{ij} c_{ij}/c(k) + U_k^{ji} c_{ji}/c(k)] = n-1$$

By theorem 3.2, we obtain the proof. □

References

[1] Doyle P.G. and Snell J.L., Random Walks and Electrical Networks, The Mathematical Association of America(1984).

[2] Tetali, P. "Random Walks and the Effective Resistance of Networks." J. Theor. Prob. 4, 101-109, 1991.

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[4] Hayt W.H. and Kemmerly J.E., Engineering Circuit Analysis,McGraw-Hill,3rd ed.(1978).