A Brief Overview of the Clairvoyant Demon

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Abstract

Let G be a connected graph, and X and Y be two tokens placed on G. We move both X and Y with simple random walks on G. Each turn one of the tokens will be moved along its walk, however we may choose which token to advance each turn. Furthermore we will assume that the entire random walk that both X and Y will take are known. The question we explore is: given the knowledge of the entire walk in advance, for what graphs G can we keep the tokens apart forever. If we can keep the tokens apart forever we will call the graph navigable. We show that if G is $\mathbb{Z}$ or a cycle then G is not navigable.

1 Introduction

Given a graph G and a token on one of the vertices, one can construct a random walk by moving the token to an adjacent vertex each turn with equal probability of going to each neighbor. A lot has been done on random walks with one token. However we can ask questions about two random walks on the same graph. We start with two tokens on a graph, and there is a demon (called the Clairvoyant Demon) who can see the entire random walk of each token. The demon must move one token each turn, but may choose which token to move. Is it possible for the demon to keep the tokens apart forever? This question was first asked in [2], and was further explored in [1]. If the demon can not see into the future (or even if he can only see finitely far into the future), then he is guaranteed to fail, as proven in [1].

As noted in [1], it is easier to solve the problem when we instead have a stronger demon called the fickle demon. The fickle demon has the power to move a token backwards along its path. Thus if the fickle demon can't keep the tokens apart, then the clairvoyant demon will also fail to keep the tokens apart. Even though the problem is harder if we only look at

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the clairvoyant demon, it has been shown that in [3] that for sufficiently large m, that K_m is navigable.

2 Preliminaries

We will denote a infinite random walk by a capital boldface letter $\mathbf{X} = v_0v_1v_2\cdots$ which will can be written as a string of symbols representing the vertices of the graph is traverses. Given a infinite random walk $\mathbf{X}$, the same letter but in lightface will represent a finite walk that terminates $\mathbf{X}$ as some vertex. $X = v_0v_1\cdots v_k$ for some $k \in \mathbb{N}$. If $X = v_0v_1\cdots v_k$, then $X^{-1} = v_kv_{k-1}\cdots v_0$ will be the reverse path of X. We will also let $< \mathbf{X}, \mathbf{Y} >$ denote the two random walks that the tokens must travel on in a given graph. We will call a pair of walks *blocked* if the demon can't move the tokens arbitrary far along their paths without them colliding at some point.

Let $U = u_0u_1\cdots u_m$ and $V = v_0v_1\cdots v_n$ be finite walks on the same graph. We say there is a *projection* from V to U, denoted $V \to U$ if there exists a map π from the set $\{0, 1, 2, \cdots, n\}$ to $\{0, 1, 2, \cdots, m\}$ such that the following hold:

1. π is surjective
2. $\pi(0) = 0$
3. $\pi n = m$
4. $|\pi(i + 1) - \pi(i)| = 1$ for all $0 \leq i < n$
5. $u_{\pi(i)} = v_i$ for all $0 \leq i \leq n$

If we have a map π that satisfies all the above except 2, we call the map a *prejection* and denote it $V \hookrightarrow U$.

Theorem 2.1. *Let X and Y be finite initial segments of $\mathbf{X}$ and $\mathbf{Y}$. if there exists a walk Z such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z^{-1}$. Then tokens that begin at $\mathbf{X}$ and $\mathbf{Y}$ will collide regardless of how they are moved, that is, $< \mathbf{X}, \mathbf{Y} >$ is blocked.*

Proof. Let $X = x_0x_1 \cdots x_m$, $Y = y_0y_1 \cdots y_n$ and $Z = z_0z_1 \cdots z_k$. Let α be a prejection from X to Z, and β be a prejection from Y to Z^{-1}. We will interpret $\beta(i)$ to be the indices of the forward walk on Z so that $\beta(0) = k$ and $\beta(n) = 0$. Assume that it is possible to prevent the tokens from colliding. Let i_t be the step that the X token is on in its walk after we have taken t turns. Define j_t similarly for Y. Observe that $\alpha(i_t) = \beta(j_t)$ means we have a collision. Let s be the minimum time such that $i_s = m$ or $j_s = n$. Thus both $\alpha(i_t)$ and $\beta(j_t)$ are defined for $0 \leq t \leq s$. Let A be the set of all t such that $\alpha(i_t) > \beta(j_t)$. Since we assumed that we never collide $s \in A$. Since α and β are prejctions, and thus only change by 1 each time, and that we only move one token at a time, by our assumption that we never collide it must be true that $t \in A$ for all $t < s$. But if $i_s = m$ then $\alpha(i_t) = 0$ for some $t < s$ but then $0 = \alpha(i_t) \not> \beta(j_t)$ since β is bounded below by 0. If $j_s = n$, then $\beta(i_t) = k$ for some $t < s$ but then $\alpha(i_t) \not> \beta(j_t) = k$ since α is bounded above by k. Thus they must collide.

□

Although we do not show it here, the converse is also true. That is, if $< \mathbf{X}, \mathbf{Y} >$ is blocked, there exists walks X, Y, and Z such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z^{-1}$. The proof of this can be found in [1].

It is important to note that there are other ways of rephrasing the problem here. As noted in [1], the problem can be rewritten as a percolation problem. Consider the infinite graph of non-negative pairs integers where an edge joins two vertices if their first or second (but not both) integer differ by a magnitude of 1. Let our pair of random walks be $\mathbf{X} = x_0x_1 \cdots$ and $\mathbf{Y} = y_0y_1 \cdots$. Starting at the origin, we label the rows of the graph with the vertices $\mathbf{Y}$ visits, and the columns of the graph with the vertices that $\mathbf{X}$ visits. Now create a new graph G by removing all vertices (x_i, y_j) where $x_i = y_j$. Then the clairvoyant demon starts at the origin and can move either to the right or up, corresponding to moving the $\mathbf{X}$ token or the $\mathbf{Y}$ token respectively, this is a directed percolation problem. For the fickle demon he can move in any direction since he can move backwards along a walk, this is a undirected percolation problem. A further analysis of the problem in this setting can be found in [1] and [3].

3 Results

We are now ready to prove the main result.

Theorem 3.1. *The infinite graph $\mathbb{Z}$ is non-navigable, that is, for any two simple random walks $\mathbf{X}$ and $\mathbf{Y}$, $< \mathbf{X}, \mathbf{Y} >$ is blocked.*

Proof. Without loss of generality we can assume that that $x_0 = -1$ and $y_0 = 1$. Let Z_k be the path from -2^k to 2^k. Let A_k be the event that the X token hits 2^k before it hits -2^k, and B_k be the event that the Y token hits -2^k before it hits 2^k. Observe that for A_t and B_s are independent for any t and s. Thus the probability of both A_1 and B_1 happening is $1/16$ in which case Z_0 blocks $< \mathbf{X}, \mathbf{Y} >$. Otherwise either both A_1 and B_1 didn't happen, or one of them did and the other didn't. If neither occurred then the probability of both A_2 and B_2 occurring is $1/16$ because we have the X token at -2 and the Y token at 2. Otherwise both tokens are now at the same point and the probability of both A_2 and B_2 occurring is $3/16$. Observe that we can induct of this process and get that the probability of A_k and B_k occurring given that $A_k \wedge B_k$ is false is at least $1/16$. Thus the probability of having a Z_k blocking $< \mathbf{X}, \mathbf{Y} >$ is at least $1/16 + 15/16(1/16) + (15/16)^2(1/16) + \cdots = (1/16)(1/(1-(15/16))) = 1$. So $\mathbb{Z}$ can't be navigated.

$\square$

We can use the above result to make statement about cycles as well.

Corollary 3.2. *Every cycle, C_n, is non-navigable.*

Proof. Observe that any random walk on a cycle can be translated into a random walk on the infinite line by having a clockwise step be a step to the right and a counterclockwise step be a step to the left. If $< \mathbf{X}, \mathbf{Y} >$ was not blocked on the cycle it would not be blocked on the line. But since the line is non-navigable, that must be false. Thus the cycle is non-navigable.

$\square$

We can also look at relationships between graphs and see how the demon does on both.

Theorem 3.3. *The graph K_n is navigable if and only if $K_{1,n}$ is navigable.*

Proof. Denote the vertices of K_n by v_i for $1 \leq i \leq n$ and the vertices of $K_{1,n}$ by u_i for $1 \leq i \leq n$ for the leaves and u_0 for the non-leaf vertex. Define φ to be a function that takes random walks on K_n to random walks on $K_{1,n}$. Let $\mathbf{X} = v_{i_1}v_{i_2}v_{i_3}\cdots$, Then $\varphi(\mathbf{X}) = u_{i_1}u_0u_{i_2}u_0u_{i_3}\cdots$. Observe that φ is a bijection between walks on K_n and walks on $K_{1,n}$. Let $<\mathbf{A},\mathbf{B}>$ be a pair of blocked paths in K_n. Then we have X, Y, and Z such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z^{-1}$. But then the pair $<\varphi(\mathbf{A}),\varphi(\mathbf{B})>$ has paths $\varphi(X), \varphi(Y)$, and $\varphi(Z)$ such that $\varphi(X) \hookrightarrow \varphi(Z)$ and $\varphi(Y) \hookrightarrow \varphi(Z^{-1})$. Thus $<\varphi(\mathbf{A}),\varphi(\mathbf{B})>$ is blocked. Similarly if $<\mathbf{A},\mathbf{B}>$ be a pair of blocked paths in $K_{1,n}$, $<\varphi^{-1}(\mathbf{A}),\varphi^{-1}(\mathbf{B})>$ is blocked in K_n. Since φ is a bijection. If K_n has a nonzero probability of picking a unblocked pair, then $K_{1,n}$ has a nonzero probability of picking a unblocked pair and vice versa. Thus K_n is navigable if and only if $K_{1,n}$ is navigable.

□

As a result of this we get

Corollary 3.4. *$K_{1,3}$, is non-navigable.*

Proof. Since every cycle is non-navigable and $K_3 = C_3$, K_3 is non-navigable. From 3.3 we know that $K_{1,3}$ is navigable if and only if K_3 is navigable. Thus $K_{1,3}$ is non-navigable.

□

4 The Clairvoyant Demon

It has been shown in [1] that the only non-navigable finite graphs are the ones we have shown to be non-navigable. Thus a graph G is navigable if and only if it is not a cycle, line segment or $K_{1,3}$. For the clairvoyant demon we can modify the techniques above to get the following.

Let $U = u_0u_1\cdots u_m$ and $V = v_0v_1\cdots v_n$ be finite walks on the same graph. We say there is a *production* from V to U if there exists a map π from the set $\{0,1,2,\cdots,n\}$ to $\{0,1,2,\cdots,m\}$ such that the following hold:

1. π is surjective
2. $\pi(0) = 0$
3. $\pi n = m$
4. $\pi(i+1) - \pi(i) \leq 1$ for all $0 \leq i < n$
5. $u_{\pi(i)} = v_i$ for all $0 \leq i \leq n$

If we have a map π that satisfies all the above except 2, we call the map a *preduction*.

It is not hard to show that a similar theorem to 2.1 holds for what we have just defined for the clairvoyant demon. However, while the theorem will produce a sufficient condition to determine if the clairvoyant demon is blocked, it will not give us a necessary condition. Thus different techniques will be needed to determine when the clairvoyant demon is blocked. Some of these techniques can be seen in [3] where it is proven that the clairvoyant demon is not blocked with nonzero probability on K_m for sufficiently large m. However these techniques depend on advance combinatorial arguments that are beyond the scope of this paper.

References

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