On the distribution of amicable numbers

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§ 1. Introduction Let $\sigma(n)$ denote the sum of the divisors of n. Two integers a, b are said to be an amicable pair if

$$\sigma(a) = \sigma(b) = a + b.$$

We say an integer n is amicable if it is a member of an amicable pair, or equivalently

$$\sigma(\sigma(n) - n) = \sigma(n).$$

Amicable numbers have a very old history dating back at least to Pythagoras who was aware that 220 and 284 form an amicable pair. We now know of more than 1100 amicable pairs [11], but it is not known if there are infinitely many.

Let $A(x)$ denote the number of amicable numbers up to x. In 1954, Kanold [9] showed that

$$A(x) < .204x$$

for all sufficiently large x. In 1955, Erdös [4] (also see [5]) proved that

$$A(x) = o(x).$$

In 1973, Rieger [13] showed that

$$A(x) = O\left(x/(\log\log\log\log x)^{\frac{1}{2}-\varepsilon}\right)$$

for every $\varepsilon > 0$. In 1975, Erdös and Rieger [6] proved that

$$A(x) = O(x/\log\log\log x).$$

In this paper we prove there is a positive constant c such that

$$(1) \qquad A(x) = O[x \cdot \exp(-c(\log\log\log x \log\log\log\log x)^{\frac{1}{2}})],$$

so that in particular,

$$A(x) = O(x/(\log\log\log x)^k)$$

for every k. It seems likely that (1) is still far from the truth about amicable numbers. In fact, by examining numerical data, Bratley, Lunnon, and McKay [1] suggest that

$$A(x) = o(x^{\frac{1}{2}}).$$

Our proof utilizes a result on primes in arithmetic progressions (see Theorem 1) and the well-known work of P. Erdös on primitive abundant numbers. We take pleasure in acknowledging a helpful conversation with Professor Erdös concerning Theorem 2.

§ 2. A note on primes in arithmetic progressions. Let k, l be integers with $k>0$ and $(k,l)=1$. It is a well-known corollary of the prime number theorem for arithmetic progressions that there is a constant $A_{k,l}$ such that (p denotes a prime)

$$S(x,k,l) \doteq \sum_{\substack{p\leqq x,\\ p\equiv l(k)}} \frac{1}{p} - \frac{1}{\varphi(k)}\log\log x = A_{k,l} + O(1/\log x).$$

We now prove a result which implies $A_{k,l}$ is bounded as k, l vary.

Theorem 1. *There is an absolute constant C such that for all $x\geqq 3$ and all k, l with $k>0$ and $(k,l)=1$, we have*

$$|S(x,k,l)|\leqq C.$$

Proof. Since

$$\sum_{\substack{0<n<x,\\ n\equiv l(k)}} \frac{1}{n} \leqq \frac{1}{k}\log x + 2,$$

the theorem holds for $x\leqq e^k$ with C any number at least 3. Now suppose $x>e^k$. From the Siegel-Walfisz theorem (see Prachar [12], Satz 8. 3, p. 144), there is an absolute constant A such that for all $t>e^k$

$$\left|\pi(t,k,l) - \frac{t}{\varphi(k)\log t}\right| < \frac{At}{\varphi(k)\log^2 t}. \tag{2}$$

By partial summation, we have

$$\begin{aligned} \sum_{\substack{p\leqq x,\\ p\equiv l(k)}} \frac{1}{p} &= \frac{1}{x}\pi(x,k,l) - \int_2^x \frac{\pi(t,k,l)}{-t^2}\,dt \\ &= \frac{1}{x}\pi(x,k,l) + \int_2^{e^k} \frac{\pi(t,k,l)}{t^2}\,dt + \int_{e^k}^x \frac{\pi(t,k,l)}{t^2}\,dt. \end{aligned} \tag{3}$$

Now by (2) we have

$$0 \leqq \frac{1}{x}\pi(x,k,l) < \frac{1}{\varphi(k)\log x} + \frac{A}{\varphi(k)\log^2 x} < \frac{1}{k\varphi(k)} + \frac{A}{k^2\varphi(k)}. \tag{4}$$

Since we trivially have $\pi(t,k,l) < 1 + \frac{t}{k}$, we have

$$0 \leqq \int_2^{e^k} \frac{\pi(t,k,l)}{t^2}\,dt < \int_2^{e^k} \left(\frac{1}{t^2} + \frac{1}{tk}\right) dt < \frac{3}{2}. \tag{5}$$

Now since

$$\int_{e^k}^x \frac{dt}{\varphi(k)\,t\log t} = \frac{1}{\varphi(k)}\log\log x - \frac{\log k}{\varphi(k)},$$

we have, using (2),

$$\left|\int_{e^k}^{x} \frac{\pi(t,k,l)}{t^2}\,dt - \frac{1}{\varphi(k)}\log\log x\right| < \frac{\log k}{\varphi(k)} + \int_{e^k}^{x} \frac{A\,dt}{\varphi(k)\,t\log^2 t} \tag{6}$$

$$< \frac{\log k}{\varphi(k)} + \frac{A}{k\varphi(k)}.$$

Hence (3), (4), (5), (6) imply

$$|S(x,k,l)| < \frac{3}{2} + \frac{\log k}{\varphi(k)} + \frac{A+1}{k\varphi(k)} + \frac{A}{k^2\varphi(k)},$$

which completes the proof of Theorem 1.

Remark 1. Using a Brun-Titchmarsh estimate in (5) we can prove

$$S(x,k,l) = \frac{1}{p(k,l)} + O\left(\frac{\log 2k}{\varphi(k)}\right)$$

where $p(k,l)$ denotes the first prime $p \equiv l(k)$ and the implied constant is uniform for all k, l, and $x \geqq k$.

Remark 2. By a similar proof, Rieger [13] obtains

$$|S(x,k,l)| = O(\log\log 3k)$$

uniformly for all $x \geqq 3$, k, l.

Theorem 2. *There is an absolute constant B such that if k, l are any integers with $(k,l)=1$, $k>0$ and if $x \geqq 3$, then the number, $M(x)$, of $n \leqq x$ for which there is no prime $p \equiv l(k)$ with $p \| n$ satisfies*

$$M(x) \leqq Bx/(\log x)^{1/\varphi(k)}. \tag{7}$$

Proof. Let $N(x)$ denote the number of $n \leqq x$ which are not divisible by any prime $p \equiv l(k)$. Then it follows from Theorem 1 and Brun's method that there is an absolute constant B' with

$$N(x) \leqq B'x/(\log x)^{1/\varphi(k)}, \quad \text{for} \quad x \geqq 3. \tag{8}$$

For a proof, one may follow the proof of Theorem 2. 3 in Halberstam and Richert [8], using Theorem 1 at the proper place. In particular, the reader should compare (8) with the second part of Corollary 2. 3. 2 in [8]. We remark that Landau [10], pp. 641—669, has an asymptotic formula for $N(x)$ (also see Delange [2], Wirsing [15], and Williams [14]). However the main advantage of (8) is the independence of the constant B' from the parameters k, l.

We now turn to the proof of (7). A positive integer f is called square full (or powerful) if for every prime p either $p \nmid f$ or $p^2 | f$. If $F(x)$ is the number of square full numbers up to x, then a result of Erdös and Szekeres [7] gives

$$F(x) \sim cx^{1/2} \tag{9}$$

where $c = \zeta(3/2)/\zeta(3)$. Now every positive integer n can be written uniquely in the form $n = mf$ where m is square free, f is square full, and $(m,f)=1$. Then the number of

$n \leqq x$ which have square full part $f > x^{1/2}$ is at most

(10) $$\sum_{f > x^{1/2}} \left[\frac{x}{f}\right] < x \sum_{f > x^{1/2}} \frac{1}{f} = O(x^{1/2}),$$

using (9). Now the number of $n \leqq x$ which have square full part $f \leqq x^{1/2}$ and for which there is no prime $p \equiv l(k)$ with $p \| n$ is at most (using (8) and assuming $x \geqq 9$)

(11) $$\sum_{f \leqq x^{1/2}} N(x/f) \leqq \sum_{f \leqq x^{1/2}} \frac{B' x}{f(\log(x/f))^{1/\varphi(k)}} \leqq \frac{B' \cdot 2^{1/\varphi(k)}}{(\log x)^{1/\varphi(k)}} \sum_{f \leqq x^{1/2}} \frac{1}{f}.$$

Hence (7) follows from (10), (11), and the fact that, by (9), the sum of the reciprocals of the square full numbers is finite.

§ 3. Abundant numbers. We note the following easy fact about the function $\sigma(n)/n$: if $m | n$ and $m \neq n$, then

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

Hence it follows that if n is abundant $(\sigma(n)/n > 2)$, then

$$a = \inf\{d : d | n,\ \sigma(d)/d \geqq 2\}$$

is such that $\sigma(a)/a \geqq 2$ and if $d | a$, $d \neq a$, then $\sigma(d)/d < 2$. Such an integer a is called primitive abundant. We have thus observed that every abundant number has a primitive abundant divisor.

If $P(x)$ denotes the number of primitive abundant numbers up to x, then a result of Erdös [3] is that

(12) $$P(x) = O[x \cdot \exp(-c_1 (\log x \log\log x)^{1/2})]$$

for some positive constant c_1. We are thus able to prove

Theorem 3. *The number of abundant numbers up to x which have no primitive abundant divisor* $\leqq y$ *is*

$$O[x \cdot \exp(-c_2 (\log y \log\log y)^{1/2})]$$

where c_2 *is a positive constant.*

Proof. If $n \leqq x$ is abundant and has no primitive abundant divisor $\leqq y$, then n has a primitive abundant divisor $> y$. If a denotes a primitive abundant number, it follows that the number of such n is at most

(13) $$\sum_{a > y} \left[\frac{x}{a}\right] < x \sum_{a > y} \frac{1}{a}.$$

Theorem 3 follows from (12) and (13) by an easy argument.

§ 4. Amicable numbers. We begin with the elementary observation that if u, v is an amicable pair with $u < v$, then u is abundant and $v = \sigma(u) - u$ is deficient $(\sigma(v)/v < 2)$. We now prove our main result:

Theorem 4. *The number of amicable numbers up to x is*

$$O[x \cdot \exp(-c_3 (\log\log\log x \log\log\log\log x)^{1/2})]$$

where c_3 *is a positive constant.*

Proof. It follows from the remark above that the number of amicable numbers up to x is at most twice the number of abundant amicable numbers up to x plus the number of perfect numbers up to x. It follows from (12) that the number of perfect numbers up to x is negligible. (Better estimates than (12) are available for the number of perfect numbers up to x, but we do not need them here.)

Suppose $n \leqq x$ is an abundant amicable number. It follows from Theorem 3 that we may assume n has a primitive abundant divisor $a \leqq (\log\log x)^{1/2}$. Since $\sigma(n) - n$ is deficient it follows that $a \nmid \sigma(n)$. Hence there is no prime $p \equiv -1\,(a)$ with $p \| n$. Thus from Theorem 2, the number of $n \leqq x$ such that $a | n$ and $a \nmid \sigma(n)$ is less than

$$\frac{B(x/a)}{(\log(x/a))^{1/a}}.$$

Hence the number of abundant amicable numbers up to x which are divisible by a primitive abundant number $a \leqq (\log\log x)^{1/2}$ is less than

$$\sum_{a \leqq (\log\log x)^{1/2}} \frac{Bx}{a(\log(x/a))^{1/a}} = O\left(\frac{x}{(\log x)^{(\log\log x)^{-1/2}}} \sum \frac{1}{a}\right) = O\left(x/\exp(\log\log x)^{1/2}\right),$$

using the fact that, by (12), the sum of the reciprocals of the primitive abundant numbers is finite. This completes the proof of Theorem 4.

We remark that with a little extra care, the constant c_3 of Theorem 4 may be taken to be $(1-\varepsilon)c_1$, for any $\varepsilon > 0$ where c_1 is the constant in (12).

Added in proof. K. K. Norton has kindly informed me that a more general formulation of Theorem 1 appears in his recent paper "On the number of restricted prime factors of an integer. I", Illinois J. Math. 20 (1976), 681—705. He also points out that the estimate (8) can be obtained without sieve methods by using certain results of Halász or Hall cited in his paper.

I wish to thank Dr. Norton for these comments and also for correcting my formulation of Remark 1.

Concerning the true order of magnitude of $A(x)$, P. Erdös in [4] and in a private communication conjectures that for each $\varepsilon > 0$ and k there is an $x_0(\varepsilon, k)$ such that

$$x^{1-\varepsilon} < A(x) < x/\log^k x \quad \text{for all} \quad x > x_0(\varepsilon, k).$$

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