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Sets on Which an Entire Function is Determined by Its Range

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Dedicated to Emil Grosswald on the occasion of his retirement

1. Introduction

Everybody knows that an entire function f is uniquely determined by the sequence of values $f(1), f(1/2), f(1/3), \ldots$. This is a special case of the so-called Identity Theorem. We shall show here that if one is given only the *set* $\{f(1), f(1/2), f(1/3), \ldots\}$ (in no particular order), then f is still determined uniquely. This result does not hold for all sequences (a_n) decreasing to zero, however; in particular we shall exhibit a sequence (a_n) of positive numbers which decreases to zero at the same rate as $(1/n)$, but such that $\{f(a_n)\} = \{g(a_n)\}$, as sets, for two different entire functions f and g. (We use the notation (a_n) for the *sequence* and $\{a_n\}$ for the *set* of points $a_1, a_2, \ldots$.)

So some sets of complex numbers are sets of range uniqueness (*sru*'s) and others are not. (We shall give a precise definition shortly.) Among other results, we shall show that sets in certain broad classes are not *sru*'s and that others are *sru*'s. We also give several examples. The notion of *sru* makes sense for sets of any infinite cardinality, but we shall consider mainly countable sets, particularly ones converging to 0. We are able to characterize the *sru*'s among many such sets.

In this article we shall use only elementary methods and results of complex variables, except in part of the next to last section. Perhaps the main surprise is that the topic has apparently been overlooked until now. There are many interesting questions about *sru*'s that we shall mention but cannot yet answer.

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2. Some Sets Which Are Not Sets of Range Uniqueness

In what follows, E will denote a set of complex numbers. For f a complex function with domain containing E let $f(E) = \{f(z)\colon z \in E\}$, the range of the restriction of f to E.

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Definition. We say that E is a *set of range uniqueness* for entire functions, or *sru*, if the following implication holds. If f and g are entire and $f(E)=g(E)$, then $f=g$.

The range uniqueness property is unchanged by alteration of a finite number of points in a set. This follows at once from

Theorem 0. *Let E be a set of range uniqueness. A set formed by adjoining to or deleting a point from E is also a set of range uniqueness.*

Proof. Let E be an *sru* and let $E'=E\setminus\{e\}$, where $e\in E$. Suppose E' is not an *sru*. Then there exist distinct entire functions f and g with $f(E')=g(E')$. Let Q and R be the polynomials

$$Q(w)=(w-f(e))(w-g(e)), \qquad R(w)=wQ(w).$$

We consider composite functions $Q\circ f(x)=Q(f(x))$, etc.

We have $Q\circ f(E)=Q\circ g(E)$ so $Q\circ f=Q\circ g$. Also, $R\circ f(E)=R\circ g(E)$, so $R\circ f=R\circ g$. It follows that

$$f=(R\circ f)/(Q\circ f)=(R\circ g)/(Q\circ g)=g$$

as meromorphic functions, and so $f\equiv g$, and E' is an *sru*.

Next, suppose that E is not an *sru*. Again, let $E'=E\setminus\{e\}$. We shall show that E' is not an *sru*. We have $f(E)=g(E)$, where f and g are distinct entire functions. Let $e'\in E'$. This time set

$$Q(w)=(w-f(e))(w-g(e))(w-f(e'))(w-g(e'))$$

and $R(w)=wQ(w)$. We have $Q\circ f(E)=Q\circ g(E)$. Also,

$$Q\circ f(E)=Q\circ f(E')\cup\{0\}=Q\circ f(E'),$$

and similarly with g in place of f. It follows that $Q\circ f(E')=Q\circ g(E')$. The same argument shows that $R\circ f(E')=R\circ g(E')$. If E' were an *sru*, then we would have $Q\circ f=Q\circ g$ and $R\circ f=R\circ g$ and so

$$f=\frac{R\circ f}{Q\circ f}=\frac{R\circ g}{Q\circ g}=g,$$

which would imply $f\equiv g$. It follows that E' is not an *sru*.

Definition. A complex function h defined on E is called *pre-entire* on E if

(a) there exists an entire function g such that $g\circ h$ extends from E to an entire function in the complex plane, and

(b) there exists some point $e\in E$ such that $g(e)\neq g(h(e))$.

In particular, if E has a finite limit point, then every non-constant entire function except the identity function is pre-entire on E. An example of a pre-entire function on $\mathbb{C}$ is $h(z)=\sqrt{z}$, with any determination of the square root, for if $g(z)=z^2$, then $g(h(z))=z$.

Theorem 1. *Let $E \subset \mathbb{C}$. Suppose that either*

(a) *E does not have a finite limit point or*

(b) *there exists a pre-entire function h on E such that the symmetric difference $h(E)\Delta E$ is finite. Then E is not a set of range uniqueness for entire functions.*

Proof. Suppose first that (a) holds. By the familiar construction of Weierstrass there is an entire function f having E as its exact set of zeros. Let $g=2f$.

To prove (b) we may now assume that E has a finite limit point. Let

$$E_0=h^{-1}\{h(E)\Delta E\}=\{e\in E: h(e)\notin E\}$$

and let $E_1=E\setminus h(E)$. (Notice that $h(E\setminus E_0)=E\cap h(E)=E\setminus E_1$.) Then $h(E_0)\cup E_1 =h(E)\Delta E$ is finite. Let $\{e_n\}\subset E$ where the e_n are distinct and $e_n\to x\in\mathbb{C}$. Finally, let g be an entire function such that $g\circ h$ extends to an entire function f and $f \neq g$ on E.

The Identity Theorem implies that $f(e_n)\neq g(e_n)$ for all large n. Moreover, since g is not constant (otherwise, $f=g$), $\{g(e_n)\}$ is an infinite set. Let $e', e^*\in\{e_n\}$ be such that $f(e^*)\neq g(e^*)\neq g(e')$ and

$$g(e'), g(e^*)\notin g(h(E)\Delta E)=g(h(E_0)\cup E_1)=f(E_0)\cup g(E_1).$$

Now construct a polynomial P such that P has roots at every point of $f(E_0)\cup g(E_1)$ as well as at $g(e')$ and at $f(e^*)$, but no root at $g(e^*)$. Then $P\circ f$, $P\circ g$ are unequal, entire, and $P\circ f(E)=P\circ g(E)$. To see the last statement, note that

$$P\circ f(E\setminus E_0)=P\circ g(h(E\setminus E_0))=P\circ g(E\setminus E_1),$$
$$P\circ f(E_0)\subset\{0\}\subset P\circ g(E),$$
$$P\circ g(E_1)\subset\{0\}\subset P\circ f(E).$$

Remark. The theorem can be strengthened a little. Indeed, instead of assuming $h(E)\Delta E$ finite, we need only assume $g(h(E)\Delta E)$ has no finite limit points (where g is entire, $g\circ h$ extends to an entire function, and $g\circ h\neq g$ on E). The only real difference in the proof is that the polynomial P is replaced by an entire function Q whose zero set is exactly $g(h(E)\Delta E)\cup\{g(e'), f(e^*)\}$. The function Q can be constructed as a Weierstrass product.

Later, in Theorem 5, we shall establish a converse of Theorem 1 for countable sets E of positive numbers having 0 as the sole limit point. Thus, the existence of a pre-entire function h on E with $h(E)\Delta E$ being a finite set completely characterizes the non-*sru*'s E among such sets. The characterization of *sru*'s for more complicated sets is an open question.

3. Examples

Examples a)–d) illustrate Theorem 1. Each set E listed here is not an *sru*. The accompanying pre-entire function h (which is, in fact, entire in each case) has $h(E)\Delta E$ finite. A corresponding entire function g is listed for which $g\neq g\circ h=f$ and $f(E)=g(E)$. The essential property of g is noted in each case.

a) $E=\{\pm 1, \pm 1/2, \pm 1/3, \ldots\}$, $h(z)=-z$, $g(z)=z$ (g not even).
b) $E=\{1/2^0, 1/2^1, 1/2^2, \ldots\}$, $h(z)=2z$, $g(z)=\sin \pi z$ $(g(2)=g(1))$.
c) $E=\{1/2^{\sqrt{2}^0}, 1/2^{\sqrt{2}^1}, 1/2^{\sqrt{2}^2}, \ldots\}$, $h(z)=z^2$,
$g(z)=(z-1/2)(z-2^{-\sqrt{2}})(z-1/4)$ $(g(1/2)=g(2^{-\sqrt{2}})=g(1/4))$.
d) $E=\{a_1, a_2, a_3, \ldots\}$, where $a_1=1$ and $a_2, a_3, \ldots$ are defined recursively as the positive solution of

$$a_n+a_n^2=a_{n-1}, \quad h(z)=z+z^2, \quad g(z)=z^2-3z \quad (g(2)=g(1)).$$

Examples A)–J) below give sets E that will be shown to be *sru*'s. Note that Theorem 1 implies that there is no pre-entire function h with $h(E)\Delta E$ finite in these cases. In each case n runs through the positive integers.

A) $\{1/n\}$	B) $\{i/n^2+1/n^3\}$
C) $\{1/2^{n!}\}$	D) $\{1/2^{(\sqrt{2}+1)^n}\}$
E) $\{1/(2^n)!\}$	F) $\{1/2^{2^n}\}\cup\{1/2^{3^n}\}$
G) $\{1/n!\}$	H) $\{1/2^n\}\cup\{1/3^n\}$
I) $\{1/p_n\}$, $p_n=n$th prime	J) $\{1/\log(n+1)\}$.

Remark. Comparison of Examples d) and A) gives a surprise, for we will now prove that the a_n of Example d) are asymptotic to the $1/n$ of Example A). Yet $\{1/n\}$ is an *sru* while $\{a_n\}$ is not. This shows that the property of being an *sru* is a delicate one.

Proposition. *Let $a_n>0$ be defined recursively by $a_1=1$, and $a_{n+1}+a_{n+1}^2=a_n$. Then $a_n \sim 1/n$.*

Proof. First it is clear that $a_n \downarrow 0$. Next, we show $a_n \geqq 1/n$. If $a_{n+1}<1/(n+1)$ holds for some positive integer n, then

$$a_n=a_{n+1}+a_{n+1}^2<\frac{1}{n+1}+\frac{1}{(n+1)^2}<\frac{1}{n},$$

and since $a_1=1$, our claim is proved by induction. Also we note that each $a_n \leqq 2/n$. For if $a_{n+1}>2/(n+1)$, then $a_n>2/(n+1)+4/(n+1)^2 \geqq 2/n$, again contradicting $a_1=1$.

Let $c_n=n a_n$. We wish to show $c_n \to 1$. We have already shown $1 \leqq c_n \leqq 2$ for all n. Let $\varepsilon>0$ be arbitrary. We shall show that for all large n, $c_n \leqq 1+\varepsilon$, thus proving the proposition. Indeed, let $n_0 \geqq 2+2/\varepsilon$ be an integer and suppose $n \geqq n_0$. If $c_n>1+\varepsilon$, then

$$c_{n-1}=(n-1)\left(\frac{c_n}{n}+\frac{c_n^2}{n^2}\right)>\frac{n-1}{n}c_n\left(1+\frac{1+\varepsilon}{n}\right)$$
$$=\frac{n^2+\varepsilon n-\varepsilon-1}{n^2}c_n>c_n+\frac{\varepsilon}{2n}.$$

In particular $c_{n-1} > 1+\varepsilon$, so that if $n-1 \geqq n_0$ we can repeat the argument and get $c_{n-2} > c_{n-1} + \varepsilon/(2n-2)$, and so on. Thus if $n \geqq n_0$ and $c_n > 1+\varepsilon$, then

$$1 \geqq c_{n_0-1} - c_n = \sum_{m=n_0}^{n} c_{m-1} - c_m > \frac{\varepsilon}{2} \sum_{m=n_0}^{n} \frac{1}{m}.$$

It follows that n cannot be arbitrarily large, which proves our assertion.

4. Sets of Range Uniqueness

In this section we establish a number of criteria which guarantee that certain sets are *sru*'s. We apply these results to show that Examples A–I are *sru*'s.

Definition. Let $\{a_n\} \subset \mathbb{C}$, $a_n \to 0$ and let h be a complex function defined on $\{a_n\}$ for all but finitely many n. We say that h *has a pole somewhere* if there exists a meromorphic function H defined on some connected neighborhood N that contains $\{a_n\}$ such that

a) $H(a_n) = h(a_n)$ for each sufficiently large n,

b) for some positive integer q the function $z \mapsto H(z^q)$ has an extension as an analytic function of z in some neighborhood of zero, and

c) $H(z)$ has a pole at some point $z_0 \in N$. (See the discussion of Examples A and B for specific functions h having a pole somewhere.)

A bounded infinite set of positive numbers E having 0 as the sole limit point is expressible as $E = \{a_n\}$ where $a_n \downarrow 0$. We call such a set *monotone*. We call a set E of complex numbers *rapidly convergent* (to zero) if $E = \{a_n\}$, where

$$|a_{n+1}| < |a_n| - B_n |a_n|^2 \tag{1}$$

for some sequence $B_n \to +\infty$.

Theorem 2. *Let $\{a_n\}$ be a set which is either monotone or rapidly convergent. Suppose there is a function h defined on $\{a_n\}$ such that exactly one of the relations $h(a_{n+1}) = a_n$ or $h(a_n) = a_{n+1}$ holds for all sufficiently large n and such that h and each iterate $h^{[2]} = h \circ h$, $h^{[3]} = h \circ h \circ h, \ldots$ has a pole somewhere. Then $\{a_n\}$ is a set of range uniqueness for entire functions.*

Example A. $\{1/n\}$ is an *sru*, for we may take $h(z) = z/(1-z)$. We have $h(1/n) = 1/(n-1)$ for $n \geqq 2$ and $h^{[k]}(z) = z/(1-kz)$ has a pole at $z = 1/k$.

Corollary (to Theorem 2). *Let φ be a polynomial which has a zero of order at least 2 at the origin. Suppose that φ is $1-1$ on some neighborhood M of $(0, 1]$. Then the set $\{\varphi(1/n)\}_{n=1}^{\infty}$ is an sru.*

Example B. $\{i n^{-2} + n^{-3}\}_{n=1}^{\infty}$ is an *sru*, for we may take $\varphi(z) = i z^2 + z^3$.

Proof of the Corollary. Let $a_n = \varphi(1/n)$. We first show that a_n satisfies (1). Write $\varphi(z) = z^b \psi(z)$, where $b \geqq 2$ and $\psi(0) \neq 0$. We have

$$\left| \left|\psi\left(\frac{1}{n}\right)\right| - \left|\psi\left(\frac{1}{n+1}\right)\right| \right| \leqq \left|\psi\left(\frac{1}{n}\right) - \psi\left(\frac{1}{n+1}\right)\right| = O(n^{-2}).$$

Thus

$$|a_n|-|a_{n+1}|=\left\{\frac{1}{n^b}-\frac{1}{(n+1)^b}\right\}\left|\psi\left(\frac{1}{n}\right)\right|+\frac{1}{(n+1)^b}\left\{|\psi\left(\frac{1}{n}\right)|-|\psi\left(\frac{1}{n+1}\right)|\right\}$$

$$\geqq\frac{b}{(n+1)^{b+1}}\left|\psi\left(\frac{1}{n}\right)\right|-\frac{c}{(n+1)^{b+2}}\geqq\frac{c'}{n^{b+1}}$$

$$\geqq B_n\left|\psi\left(\frac{1}{n}\right)\right|^2 n^{-2b}=B_n|a_n|^2$$

with $B_n=c''\,n^{b-1}$.

We require an inverse of φ for our argument; we indicate briefly how this can be achieved.

Puiseux Series Representation. Suppose that $\sum a_n z^n$ *converges in some neighborhood of the origin and the coefficients* $a_0, \ldots, a_{p-1}$ *all vanish, but* $a_p \neq 0$. *Then the equation* $w=\sum a_n z^n$ *has a solution* $z=\sum c_n t^n$, *convergent in some neighborhood of* $t=0$, *where* $t=w^{1/p}$. *The coefficients* c_n *may be found by formal power series manipulations.* [1, vol. III, §63, p. 246]

Returning to the proof of the Corollary, we note that φ is $1-1$ on M. Let φ^{-1} be defined on $N=\varphi(M)$ as that branch mapping onto M. We define h on N by

$$h(z)=\varphi(\varphi^{-1}(z)/\{1-\varphi^{-1}(z)\}).$$

Then $h(a_n)=a_{n-1}$ for all $n\geqq$ some L. Also, for $k\geqq 1$,

$$h^{[k]}(z)=\varphi(\varphi^{-1}(z)/\{1-k\,\varphi^{-1}(z)\}),$$

and this function has a pole at $z=\varphi(1/k)\in N$. Also, letting $H(z)$ denote the function $h^{[k]}(z)$, we note that $z\mapsto H(z^q)$ has an analytic extension to a neighborhood of zero for a suitable positive integer q. This follows from the Puiseux Series Representation. Thus, by Theorem 2, $\{\varphi(1/n)\}$ is an *sru*.

The proof of Theorem 2 and most of the subsequent criteria for *sru*'s depend on the following two lemmas.

Lemma 1. *Let* f *be a non constant function which is analytic at* 0 *and vanishes there. Let* $\{a_n\}$ *be a monotone or rapidly convergent set. Then* $|f(a_{n+1})|<|f(a_n)|$ *for all sufficiently large* n.

Proof of Lemma 1. Suppose $\{a_n\}$ is monotone. Let $F(z)=f(z)\overline{f(\bar{z})}$, so that F is analytic at 0 and non negative on some line segment $[0,b]$. Because $f\not\equiv 0$, neither is F. It follows that $F'(x)$ does not vanish infinitely often on $[0,b]$. Thus $|F(a_n)|=|f(a_n)|^2$ is strictly decreasing for all sufficiently large n.

Next suppose that $\{a_n\}$ is rapidly convergent. Write $f(z)=\sum_{i\geqq I} c_i z^i$, with I the least integer for which $c_I\neq 0$. Then for all sufficiently large integers n we have

$$\left|\frac{f(a_{n+1})}{f(a_n)}\right| = \left|\frac{c_I a_{n+1}^I + O(|a_{n+1}|^{I+1})}{c_I a_n^I + O(|a_n|^{I+1})}\right|$$
$$\leqq \left|\frac{a_{n+1}}{a_n}\right|^I + B|a_n|$$
$$\leqq \left|\frac{a_{n+1}}{a_n}\right| + B|a_n| < 1.$$

Lemma 2. *Let $\{a_n\}$ be a convergent sequence of complex numbers and let f and g be different entire functions satisfying* (i) $\{f(a_n)\} = \{g(a_n)\}$ *(as sets) and* (ii) *both* $\{|f(a_n)|\}$ and $\{|g(a_n)|\}$ *are ultimately monotone decreasing sequences. Then there exists an integer $k \neq 0$ such that $f(a_n) = g(a_{n+k})$ holds for all sufficiently large integers n.*

Proof of Lemma 2. Let $k(n)$ be an integer such that $f(a_n) = g(a_{n+k(n)})$. Since $|f(a_n)|$, $|g(a_n)|$ are both eventually strictly decreasing, we have $n + k(n) \to \infty$ as $n \to \infty$. Moreover, we have an n_1 such that if $n \geqq n_1$ then $|f(a_m)| > |f(a_n)| > 0$ for all $m < n$ with $f(a_m) \neq 0$ and $|g(a_l)| > |g(a_n)| > 0$ for all $l < n$ with $g(a_l) \neq 0$. If n is large enough, we have $n \geqq n_1$ and $n + k(n) \geqq n_1$. Then

$$\dots < |f(a_{n+2})| < |f(a_{n+1})| < |f(a_n)| = |g(a_{n+k(n)})|$$
$$> |g(a_{n+1+k(n)})| > |g(a_{n+2+k(n)})| > \dots .$$

But $f(a_{n+1}) = g(a_{n+1+k(n+1)})$. Let $m = n + 1 + k(n+1)$. Since $n + k(n) \geqq n_1$ and $|g(a_m)| < |g(a_{n+k(n)})|$, we must have $m > n + k(n)$. If i is such that $f(a_i) = g(a_{n+1+k(n)})$, since $|f(a_i)| < |f(a_n)|$ and $n \geqq n_1$, we must have $i > n$. Thus

$$|f(a_i)| \leqq |f(a_{n+1})| = |g(a_m)| \leqq |g(a_{n+1+k(n)})| = |f(a_i)|,$$

so that equality must hold at each point. Thus $m = n + 1 + k(n)$, i.e., $k(n+1) = k(n)$. Hence there is an integer k for which $k(n) = k$ for all large n. Since $f \neq g$, we have $k \neq 0$.

Proof of Theorem 2. We suppose that f and g are entire, $f \neq g$, and $f(\{a_n\}) = g(\{a_n\})$. Moreover, we may assume that f and g are both non constant. By subtracting a suitable constant, we may assume $f(0) = g(0) = 0$. This is so because, by continuity $f(0)$ is the sole limit point of $f(\{a_n\})$, $g(0)$ is the sole limit point of $g(\{a_n\})$ and $f(\{a_n\}) = g(\{a_n\})$. By the lemmas, possibly with the roles of f and g interchanged, there is an integer $k > 0$ such that $f(a_n) = g(a_{n+k})$ for all large n.

Suppose that $h(a_n) = a_{n+1}$ for all large n. (The case $h(a_n) = a_{n-1}$ for all large n is treated analogously.) Let $F = g \circ h^{[k]}$. Then, for all large n we have

$$F(a_n) = g(h^{[k]}(a_n)) = g(a_{n+k}) = f(a_n).$$

Thus F has f as an entire extension from $\{a_n\}_{n \geq L}$, for some L. Let H be the meromorphic function postulated by the condition that $h^{[k]}$ has a pole somewhere. In particular, $H(a_n) = h^{[k]}(a_n)$ for $n = L, L+1, \dots$. Consider the two functions $f(z^q)$ and $g(H(z^q))$, where q is an integer for which $H(z^q)$ extends as an analytic function of z in a connected neighborhood Ω of 0. These two functions are both analytic on Ω, and since they agree on a sequence $(a_n^{1/q})_{n \geqq L}$ that approaches 0 (where any q-th root of a_n may be used), they must agree throughout Ω by the Identity Theorem. (We have in effect proved a version of the Identity Theorem for functions with branch singularities at the limit point.)

Since $f(z^q) = g(H(z^q))$ in Ω, this equality persists throughout $N^{1/q}$, where N is the connected and simply connected neighborhood of $\{a_n\}_{n \geqq L}$ where H is supposed meromorphic. (Any branch of the q-th root may be taken.) Now $z \mapsto f(z^q)$ is bounded in a neighborhood of $p^{1/q}$, where p is the pole of H. But by Liouville's Theorem $z \mapsto g(H(z^q))$ is unbounded in each deleted neighborhood of $p^{1/q}$. This contradiction establishes the result.

Theorem 3. *Suppose that $\{a_n\}$ is either monotone or rapidly convergent and is not a set of range uniqueness for entire functions. Then there is an integer $k > 0$ such that both*

$$\lim_{n \to \infty} \frac{a_{n+k}}{a_n}, \qquad \lim_{n \to \infty} \frac{\log|a_{n+k}|}{\log|a_n|},$$

exist and are finite. Moreover, the second limit is rational.

Proof. Assume $\{a_n\}$ is not an *sru*. Then there are unequal entire functions f, g with $f(\{a_n\}) = g(\{a_n\})$. As before, we may assume $f(0) = g(0) = 0$. By the lemmas and possibly interchanging f and g, there is a positive integer k such that $f(a_n) = g(a_{n+k})$ for all large n. Let l, m be the smallest positive integers with $f^{(l)}(0)/l! = c \neq 0$, $g^{(m)}(0)/m! = d \neq 0$. We then have

$$f(z) \sim c z^l, \quad g(z) \sim d z^m \quad \text{as } z \to 0.$$

Thus

$$c a_n^l \sim d a_{n+k}^m \quad \text{as } n \to \infty, \tag{2}$$

so that

$$\frac{a_{n+k}}{a_n} \sim \left(\frac{c}{d}\right)^{1/m} a_n^{(l/m)-1} \quad \text{as } n \to \infty.$$

Thus $l \geqq m$ (since a_{n+k}/a_n is bounded) and

$$\lim_{n \to \infty} \frac{a_{n+k}}{a_n} = \begin{cases} 0, & \text{if } l > m \\ \left(\dfrac{c}{d}\right)^{1/m}, & \text{if } l = m. \end{cases}$$

Moreover, $\log|c a_n^l| - \log|d a_{n+k}^m| \to 0$, so that

$$\lim_{n \to \infty} \frac{\log|a_{n+k}|}{\log|a_n|} = \frac{l}{m}, \tag{3}$$

a rational number.

Remark. We can replace (2) with the stronger assertion

$$c a_n^l - d a_{n+k}^m = O(|a_n|^{m+1}).$$

Moreover, in place of (3) we have the stronger relation

$$\frac{\log|a_{n+k}|}{\log|a_n|}=\frac{l}{m}+O\left(\frac{1}{\log|a_n|}\right).$$

An example where Theorem 3 does not apply is provided by the set

$$\{\pm 2^{-n}\}\cup\{\pm 3^{-n}\}, \qquad n=1,2,3,\ldots.$$

In any enumeration as a sequence (a_n) with $|a_n|$ non-increasing, the ratio a_{n+k}/a_n does not converge to a limit as $n\to\infty$ for any $k\neq 0$. Yet this set is not an *sru*, for we can take g as any non even entire function and $f(z)=g(-z)$.

We now use Theorem 3 and the above remark to establish $\{a_n\}$ as an *sru* for Examples C)–I).

If $E=\{a_n\}$, $a_n\downarrow 0$, and $\log a_{n+1}/\log a_n\to\infty$, then Theorem 3 implies E is an *sru*. This establishes Example C) $(E=\{1/2^{n!}\})$ as an *sru*. Similarly if $\log a_{n+1}/\log a_n\to r$ and no integral power of r is rational, then E is an *sru*. Thus Example D) $(E=\{1/2^{(\sqrt{2}+1)^n}\})$ is an *sru*. (cf. Example c) Example F) $(E=\{1/2^{2^n}\}\cup\{1/2^{3^n}\})$ is an *sru* for if E is rewritten as $\{a_n\}$ where $a_n\downarrow 0$, then for every $k>0$, $\log a_{n+k}/\log a_n$ does not tend to a limit.

For example E) $(E=\{1/(2^n)!\})$ note that

$$\frac{\log a_{n+k}}{\log a_n}=2^k+\frac{k\cdot 2^k}{n}+O\left(\frac{1}{n^2}\right)$$

so that E is an *sru* by the above remark.

For Example H) $(E=\{1/2^n\}\cup\{1/3^n\}=\{a_n\})$, note that for each $k>0$, a_{n+k}/a_n does not tend to a limit, so the theorem implies E is an *sru*.

Finally, for Example I) $(E=\{1/p_n\}$, $p_n=n$-th prime) we use the fact that for each k, $p_{n+k}\sim p_n$, which is a corollary of the prime number theorem. Then if $\{1/p_n\}$ is not an *sru*, (2) implies $l=m$ and $c=d$. Thus the remark implies $p_n^{-l}-p_{n+k}^{-l}=p_n^{-l-1}\cdot O(1)$. But

$$\frac{1}{p_n^l}-\frac{1}{p_{n+k}^l}=\frac{(p_{n+k}-p_n)(p_{n+k}^{l-1}+p_{n+k}^{l-2}p_n+\ldots+p_n^{l-1})}{p_n^l p_{n+k}^l}$$
$$\sim\frac{l(p_{n+k}-p_n)}{p_n^{l+1}}.$$

Thus we would have $p_{n+k}-p_n=O(1)$, contradicting the elementary fact that there are arbitrarily large gaps in the sequence of primes. Thus $\{1/p_n\}$ is an *sru*.

Theorem 4. *Suppose that* $a_n\downarrow 0$ *and there is a positive integer* k *such that* $\lim(a_{n+k}/a_n)=0$ *and* $\lim(\log a_{n+k}/\log a_n)=1$. *Then* $\{a_n\}$ *is a set of range uniqueness for entire functions.*

Proof. Suppose not. By Theorem 3, there is an integer $k_0>0$ such that $\lim(a_{n+k_0}/a_n)=r$, $\lim(\log a_{n+k_0}/\log a_n)=s$. Then

$$\lim\frac{a_{n+kk_0}}{a_n}=r^k,\qquad \lim\frac{\log a_{n+kk_0}}{\log a_n}=s^k.$$

By our hypothesis, we see that $r^k=0, s^k=1$, so that $r=0$, $s=1$. By the proof of Theorem 3, we see that $r=0$ implies $s>1$, a contradiction.

Remark. This theorem establishes Example G) ($E=\{1/n!\}$) as an *sru*.

We now establish a strong converse of Theorem 1 in a special case.

Theorem 5. *Suppose $a_n \downarrow 0$ and $E=\{a_n\}$ is not a set of range uniqueness for entire functions. Then there is a pre-entire function h on E such that $h(E)\Delta E$ is finite. In fact, there is an integer $k \neq 0$ such that $h(a_n)=a_{n+k}$ for all large n. Moreover, h can be chosen so that it has an analytic extension to a neighborhood of the open interval $(0,\varepsilon)$ for some $\varepsilon>0$.*

Proof. Let f, g be unequal entire functions with $f(E)=g(E)$. As in the proof of Lemma 1, we may assume $f(0)=g(0)=0$. Let $F(z)=f(z)\overline{f(\bar{z})}$, $G(z)=g(z)\overline{g(\bar{z})}$. Then F, G are entire, non negative on the real axis, and by Lemma 2, applied to f and g, there is an integer $k \neq 0$ with $F(a_n)=G(a_{n+k})$ for all large n. Since G is not identically 0, there is a $\delta>0$ such that G is monotone increasing on $(0,\delta)$. Thus $F \neq G$.

By choosing δ sufficiently small we may use the Puiseux Series Representation to define G^{-1} as an analytic function on some neighborhood U of $(0, G(\delta)]$.

There is an $\varepsilon>0$ and a neighborhood N of $(0,\varepsilon)$ such that $F(N)\subset U$. Let $h=G^{-1}\circ F$. We thus have h well-defined and analytic on N. Also, $G\circ h=F \neq G$ on E, so h is pre-entire. Finally, $h(a_n)=a_{n+k}$ for all large n, so that $h(E)\Delta E$ is finite, and the theorem is proved.

Using Theorem 5, we now establish the following, perhaps surprising, result.

Theorem 6. *If $a_n \downarrow 0$, then there is a set of range uniqueness for entire functions $\{b_n\}$ with $b_n \downarrow 0$ and $b_n \sim a_n$.*

Proof. We may assume $\{a_n\}$ is not an *sru*, for otherwise take $b_n=a_n$. Let $\varepsilon_n \downarrow 0$ be any sequence such that for each n, $a_n > a_{n+1}(1+\varepsilon_{n+1})$. Let

$$b_n=\begin{cases} a_n, & \text{if } n \text{ is not a square,} \\ a_n(1+\varepsilon_n), & \text{if } n \text{ is a square.} \end{cases}$$

It is obvious that $b_n \downarrow 0$ and $b_n \sim a_n$. It remains to show that $\{b_n\}$ is an *sru*. Suppose it is not. By Theorem 5, there is an $\varepsilon>0$, functions h_1, h_2 analytic on a neighborhood of $(0,\varepsilon)$, and positive integers k_1, k_2 with $h_1(a_n)=a_{n+k_1}$, $h_2(b_n)=b_{n+k_2}$ for all large n. Thus letting $h_1(0)=h_2(0)=0$, we have h_1, h_2 continuous on $[0,\varepsilon)$. Consequently there is a $\delta>0$ such that the iterates $h_1^{[k_2]}, h_2^{[k_1]}$ are defined and analytic in some neighborhood of $(0,\delta)$.

There are infintely many n such that $n, n+k_1 k_2$ are both not squares. If n is large and such a number, then

$$b_n=a_n, b_{n+k_1k_2}=a_{n+k_1k_2} \quad \text{and} \quad h_1^{[k_2]}(a_n)=a_{n+k_1k_2}=h_2^{[k_1]}(a_n).$$

We shall prove below that $h_1^{[k_2]}=h_2^{[k_1]}$. But also there are infinitely many non-squares n for which $n+k_1k_2$ is a square. If n is large and such a number, we have

$$a_{n+k_1k_2}=h_1^{[k_2]}(a_n)=h_2^{[k_1]}(a_n)=h_2^{[k_1]}(b_n)=b_{n+k_1k_2},$$

contradicting the definition of the sequence $\{b_n\}$.

It remains to prove that $h_1^{[k_2]}=h_2^{[k_1]}$. Changing notation, we have functions H_1 and H_2, each meromorphic on a neighborhood N of $(0,\varepsilon]$. Furthermore, we have entire functions f_1, f_2 and g_1, g_2 so that $f_1(0)=f_2(0)=g_1(0)=g_2(0)=0$ and such that $f_1(H_1(z))=g_1(z)$ and $f_2(H_2(z))=g_2(z)$ on N. Moreover, we have a sequence $\alpha_n\downarrow 0$ in $(0,\varepsilon]$ such that $H_1(\alpha_n)=H_2(\alpha_n)$ for $n=1,2,\ldots$. We must prove that then $H_1=H_2$ on N. This is just a version of the Identity Theorem for functions analytic except for a branch singularity. By the Puiseux representation there is a positive integer q such that $H_1(z)$ and $H_2(z)$ are represented by series of powers of $z^{1/q}$. More precisely, there are functions $\tilde{H}_1$ and $\tilde{H}_2$ analytic in a neighborhood M of 0, and an integer $q\geqq 1$ such that

$$f_1(\tilde{H}_1(t^q))=g_1(t^q); \quad f_2(\tilde{H}_2(t^q))=g_2(t^q).$$

Furthermore, there are only finitely many such pairs $\tilde{H}_1$, $\tilde{H}_2$. Since

$$f_1(H_1(t^q))=g_1(t^q); \quad f_2(H_2(t^q))=g_2(t^q),$$

we see that $H_1(t^q)$ and $H_2(t^q)$ have analytic extensions $K_1(t)$ and $K_2(t)$ respectively, to a neighborhood M^* of 0. But, with, say, the positive determination of the q-th roots, $K_1(\alpha_n^{1/q})=H_1(\alpha_n)=H_2(\alpha_n)=K_2(\alpha_n^{1/q})$ for $n=1,2,\ldots$. So by the Identity Theorem, we have $K_1=K_2$. That is, $\tilde{H}_1(t^q)=\tilde{H}_2(t^q)$. So $H_1(t^q)=H_2(t^q)$ on a neighborhood of $(0,\varepsilon^{1/q}]$. Hence $H_1=H_2$ on $(0,\varepsilon]$ and consequently $H_1=H_2$ on N. This completes the proof of Theorem 6.

5. Generalizations of Range Uniqueness

We now define a notion of weak set of range uniquness where cardinalities are taken into account.

Definition. If f is a complex function, $E\subset\mathbb{C}$, and $\alpha\in\mathbb{C}$, let $N(f,E,\alpha)$ denote the cardinality of the set $\{e\in E: f(e)=\alpha\}$. (We take no account of multiplicity here.) If g is also a complex function and if $N(f,E,\alpha)=N(g,E,\alpha)$ for all α, we write $f(E)\equiv g(E)$. We say E is a *weak set of range uniqueness* for entire functions, or *wsru*, if $f(E)\equiv g(E)$ for entire functions f, g implies $f=g$.

Thus if E is an *sru*, then E is a *wsru*. The converse is not true as we now see.

Theorem 7. *If $a_n\downarrow 0$, then $\{a_n\}$ is a weak set of range uniqueness for entire functions.*

This theorem is an immediate corollary of the lemmas. We leave the few details for the interested reader to supply.

Example a) of a non-*sru* is also an example of a non-*wsru*. Another simple example is patterned after Example b). Let $E=\{2^n: n \text{ an integer}\}$, $g(z)=\sin\pi z$, $f(z)=\sin 2\pi z$. Note that $N(f,E,0)=N(g,E,0)=\aleph_0$.

Definition. Suppose that E is a set with the following property: whenever $f(E) = g(E)$ for entire functions f and g, then the first $r+1$ Maclaurin coefficients of f and g coincide. We call such a set E a *partial sru of degree r*. (A partial *sru* of degree ∞ is an *sru*.) Here we shall show that a monotone set E with terms converging sufficiently slowly to zero is a partial *sru* of some degree r.

Theorem 8. *Suppose that $\{a_n\}$ satisfies $a_n \downarrow 0$ and $a_n - a_{n+1} = o(a_n^r)$ for some positive integer r. Then $\{a_n\}$ is a partial sru of degree r.*

Proof. Suppose that f and g are entire functions with $f(\{a_n\}) = g(\{a_n\})$. Arguing as in § 4 we may assume $f(0) = g(0) = 0$ and that there is some integer k such that $f(a_n) = g(a_{n+k})$ holds for all sufficiently large n. If $k=0$. then $f=g$ by the Identity Theorem. Without loss of generality we may assume that $k>0$. If $k>1$ note that

$$a_n - a_{n+k} = o(a_n^r) + \ldots + o(a_{n+k-1}^r) = o(a_n^r).$$

We have

$$g(a_n) - f(a_n) = \int_{a_{n+k}}^{a_n} g'(u)\, du = O(a_n - a_{n+k}) = o(a_n^r).$$

It follows that the first $r+1$ Maclaurin coefficients of f and g agree and so $\{a_n\}$ is a partial *sru* of degree r.

Example. Let $a_n = (\log(n+7))(n+7)^{-1/2}$. We have

$$0 < a_n - a_{n+1} = O\left(\frac{\log(n+7)}{(n+7)^{3/2}}\right) = o(a_n^3).$$

and so $\{a_n\}$ is a partial *sru* of degree 3.

Example J. For $a_n = 1/\log(n+1)$ we have

$$0 < a_n - a_{n+1} < \frac{1}{(n+1)\log^2(n+1)} = o(a_n^r)$$

for any fixed positive integer r as $n \to \infty$. Thus $\{1/\log(n+1)\}$ is an *sru*.

Example d) $(a_1 = 1, a_n + a_n^2 = a_{n-1})$ is a partial *sru* of exact degree 1. Indeed Theorem 8 guarantees that $\{a_n\}$ is of degree 1 and the pair of functions $g(z) = 3z - z^2$, $f(z) = 3(z+z^2) - (z+z^2)^2$, which satisfy $\{f(a_n)\} = \{g(a_n)\}$, differ in the coefficients of z^2.

6. Sets of Range Uniqueness for Functions Analytic on Subsets of the Plane

In this section, we give two instances in which a set E is not a set of range uniqueness for functions analytic on some subset G of the complex plane, but is a set of range uniqueness for functions analytic on a slightly larger set. We thus see that the property of E being an *sru* on G can depend on the shape of G far away from E.

Definition. For two sets K, L of complex numbers with $K \subset L$ we say that K is a set of range uniqueness for L whenever the following implication holds: if f and g are holomorphic functions on L and if $f(K)=g(K)$, then $f=g$.

The case we have studied so far is $L=\mathbb{C}$. We make a brief study of some other domains L.

Theorem 9. *Let* $\mathbb{D}=\{z:|z|<1\}$ *be the unit disc. There is a sequence* $a_n \uparrow 0$, $\{a_n\} \subset \mathbb{D}$, *that is not a set of range uniqueness for* $\mathbb{D}$, *but that is a set of range uniqueness for every region* G *such that* $\mathbb{D} \cup \{1\} \subset G$.

Remark. The point of this theorem is that the property of $\{a_n\}$ being an *sru* for a region G depends on properties of ∂G far away from the sole limit point of $\{a_n\}$. This is a kind of "action at a distance."

Proof. Let Ω be a Jordan region with boundary J that lies in $\mathbb{D}$ except that $1 \in J$. We further require that $0 \in \Omega$, that Ω is symmetric about the real axis, and that J has an exponential cusp at $z=1$ (more about this later).

Let h be the Riemann map of $\mathbb{D}$ onto Ω so that $h(0)=0$, $h'(0)>0$. Note that by a theorem of Carathéodory [2, Vol. III, Corollary to Theorem 2.24, p. 70] h extends continuously as a $1-1$ function to $\overline{\mathbb{D}}$ and maps ∂D onto J. Since $h(z) = \overline{(h(\bar{z}))}$ by the uniqueness of the Riemann map, we have $h(1)=1$.

Note that h is real on $[-1, 1]$. By Schwarz's Lemma, since $|h(z)|<|z|$ for $z \in \mathbb{D} \setminus \{0\}$ and since $h(1)=1$, we have $h(x)<x$ for $0<x<1$ and $h(x)>x$ for $-1<x<0$. Also, the only fixed point of h in $\mathbb{D}$ is $z=0$. Choose a_1 with $-1<a_1<0$, and define $\{a_n\}$ recursively by $h(a_n)=a_{n+1}$. We have $a_n \uparrow 0$ because 0 is the only fixed point of h.

By Theorem 1 adapted to $\mathbb{D}$, $\{a_n\}$ is not a set of range uniqueness for functions analytic in $\mathbb{D}$, because $h(\{a_n\}) \Delta \{a_n\} = \{a_1\}$ is finite.

Now let us prove that $\{a_n\}$ is an *sru* for G if G satisfies our hypotheses. By the lemmas of §4, adapted to G, if f and g are analytic in G with $f(\{a_n\})=g(\{a_n\})$ and $f \neq g$, then there is an integer $l \neq 0$ (say $l>0$ by interchanging the roles of f and g if necessary) such that $f(a_{n+l})=g(a_n)$ for all sufficiently large n. Now $f(h^{[l]}(z))=g(z)$ for all $z \in \mathbb{D}$ because it is true for all $z \in \{a_n\}$, $n \geqq n_0$, and this last set has a limit point in $\mathbb{D}$. Without loss of generality, take $g(1)=0$. Since as $x \to 1^-$, $h^{[l]}(x) \to 1$, we also have $f(1)=0$. Because f and g are analytic at $z=1$, we have $f(x) \sim c(1-x)^p$ and $g(x) \sim d(1-x)^q$, where $c, d \neq 0$ and p and q are positive integers. Hence

$$c(1-h^{[l]}(x))^p \sim d(1-x)^q$$

or

$$\frac{1-h^{[l]}(x)}{(1-x)^\alpha} \to e \neq 0, \infty \quad (x \to 1^-)$$

where $\alpha=q/p$ and $e=(d/c)^{1/p}$.

By the Poisson integral formula,

$$\begin{aligned}[1-h(x)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[1-h(e^{i\theta})]\,\frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta \\ &\geqq \frac{1}{2\pi}\int_{1-x}^{2(1-x)}[1-|h(e^{i\theta})|]\frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta \\ &\geqq [1-|h(e^{i(1-x)})|]\int_{1-x}^{2(1-x)}\frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta \\ &\geqq c[1-|h(e^{i(1-x)})|]\end{aligned}$$

for a suitable constant $c>0$, as we see on estimating the last integrand. We have used above a monotonicity hypothesis on $|h(e^{i\theta})|$ for small θ. Indeed, let us now suppose that $1-|h(e^{i\theta})|=\dfrac{1}{\log\frac{1}{|\theta|}}$ for small θ. We then derive

$$[1-h(x)]\geqq c/\log\frac{1}{1-x}$$

so that

$$\frac{[1-h(x)]}{(1-x)^{\alpha}}\to\infty \quad \text{as } x\to 1- \quad \text{for any } \alpha>0.$$

However for $0<x<1$, $h(x)<x$, $h(h(x))<h(x)$, …, so that for $l=2,3,\dots$,

$$\frac{1-h^{[l]}(x)}{(1-x)^{\alpha}}>\frac{1-h(x)}{(1-x)^{\alpha}}\to\infty.$$

This contradiction shows that no such f and g exist, so that $\{a_n\}$ must be a set of range uniqueness for functions holomorphic in G. The proof of the theorem is complete.

The next result actually proves more in a shorter space. The proof is a slight variation on ideas in the earlier part of the paper.

Theorem 10. *Let $E=\{1,\frac{1}{2},\frac{1}{3},\dots\}$, and let $G=\{z:|z|<2\}$. Then E is not a set of range uniqueness for functions analytic on G, but it is a set of range uniqueness for functions bounded on G near $z=2$ and analytic on G.*

Remark. By Theorem 0, adapted to the present context, we can replace E in the above statement by any $E_n=\left\{\frac{1}{n},\frac{1}{n+1},\frac{1}{n+2},\dots\right\}$.

Proof of the Theorem. First, let

$$f(z)=\frac{(z-1)(z-\frac{1}{2})}{(z-2)^2},\qquad g(z)=f\left(\frac{z}{z+1}\right).$$

Then $f(E)=g(E)$, f and g are analytic on G, and $f\neq g$, since $f(-1/2)\neq g(-1/2)$, say. This shows that E is not an *sru* for G.

Now suppose that f and g are bounded functions on G near $z=2$, analytic on G and $f\neq g$, but $f(E)=g(E)$. We will arrive at a contradiction. By Lemmas 1 and 2, there is an integer $k>0$ such that $f(1/n)=g(1/(n-k))$ for all large n (after possibly interchanging f and g). Thus

$$g(z)=f\left(\frac{z}{kz+1}\right) \tag{4}$$

when $z=1/n$ and n is large enough. Hence (4) holds identically for all z in a neighborhood of 0. But the left side is defined whenever $|z|<2$. So we may use (4) to analytically continue f to the set

$$\left\{\frac{z}{kz+1}: |z|<2\right\}\cup\{z: |z|<2\},$$

which is the whole Riemann sphere $\mathbb{C}^\wedge$ when $k>1$, and is $\mathbb{C}^\wedge\setminus\{2\}$ when $k=1$. In case $k>1$, f is therefore analytic on $\mathbb{C}^\wedge$ and hence constant, so we may suppose $k=1$. Now $f(z)$ is bounded as $z\to 2$ within $\{|z|<2\}$ and it follows from (4) that $f(z)$ is bounded as $z\to 2$ within $\{z/(z+1): |z|<2\}$. (Note that this set is the exterior of the closed disc whose diameter is $[2/3, 2]$.) Since the union of these two sets contains a deleted neighborhood of $z=2$, it follows that f is bounded in this deleted neighborhood. Hence f is analytic on $\mathbb{C}^\wedge$ and is therefore a constant. Since this is impossible, the theorem is proved by contradiction.

Remarks. Essentially the same proof shows that we may replace the boundedness of $f(z)$ at $z=2$ by the assertion that $f(z)=o(|z-2|^{-1})$ as $z\to 2$ within G. This could be useful in discussing sets of range uniqueness for the Hardy classes H^p. Also, an examination of the proof will show that we have actually proved the following. If f and g are distinct analytic functions on G for which $f(E)=g(E)$, then at least one of f, g is unbounded in G near $z=2$. Moreover, if g is bounded in G near $z=2$, then $g(z)=f(z/(z+1))$.

7. Problems

Some of the following problems probably will turn out to be easy and some hard, but we do not know which. Questions asked for *sru*'s often continue to make sense (and may be easier) when asked about *wsru*'s or partial *sru*'s of finite degree.

Problem 1. Can an arc be an *sru*? How about an analytic arc?

Problem 2. Can a perfect set be an *sru*?

Problem 3. Which triangles are *sru*'s? (By triangle we could mean either the perimeter or the whole figure.) Equilateral triangles are obviously not *sru*'s. How about a $3-4-5$ triangle?

Problem 4. Can an open set be an *sru*?

Problem 5. Characterize all countable *sru*'s. It should be remarked that a countable set of real numbers that is dense on $\mathbb{R}$ cannot be an *sru*. This follows from the main theorem of [3].

Problem 6. Characterize those sequences $b_1 < b_2 < \ldots$ of positive integers for which $\{1/b_n\}$ is not an *sru*. For example, it follows from the methods of this paper, that $\{b_n\}$ has asymptotic density 0 in the natural numbers (that is, $b_n/n \to \infty$). In fact, we can show that there is some $\alpha > 1$ such that $b_n > \alpha^n$ for all large n. If $\{b_n\}$ also has the property that $\{b_n^{1/n}\}$ is bounded, must it be true that for some integers k, r we have $\{b_n\}$ (modulo finite sets) the union of k geometric progressions each with common ratio r?

Problem 7. What are the maps $\varphi: \mathbb{C} \to \mathbb{C}$ such that if E is an *sru*, then $\varphi(E)$ is an *sru*?

Problem 8. Let $\mathscr{E}^*$ be a class of entire functions, say the class of entire functions of exponential type, or those of type $< \pi$. In an obvious way, one can define the notion of sets of range uniqueness relative to $\mathscr{E}^*$. Study this situation. For example, taking $\mathscr{E}^*$ to be $\mathscr{P}$, the class of polynomials, one can show that $E = \{2^{2^{2^n}}\}$ is an *sru*. This is a consequence of Theorem 3 after the transformation $p(z) \to 1/p(1/z)$.

Problem 9. What conditions upon a set $\{a_n\}$ weaker than those of Theorems 2 and 3, suffice to guarantee that $\{a_n\}$ is an *sru*? For example, do Theorems 2 and 3 remain true if the monotone or rapidly convergent condition is replaced by the pair of conditions $a_n \to 0$ and $\arg a_n \to 0$?

Problem 10. Can an uncountable set be an *sru*?

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