Divisors of the Middle Binomial Coefficient

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Abstract. We study some old and new problems involving divisors of the middle binomial coefficient $\binom{2n}{n}$.

1. INTRODUCTION. In the center of the $2n$th row of Pascal's triangle we meet the maximal entry $\binom{2n}{n}$. These middle binomial coefficients have a rich history. For example, the fact that $\binom{2n}{n}$ is divisible by the product of the primes in the interval $(n, 2n)$ was exploited by Chebyshev in 1850 to obtain upper bounds and lower bounds for the distribution of primes. These bounds were so good that it seemed a promising path to the prime number theorem, but eventually that goal was reached by other methods.

The central binomial coefficient $\binom{2n}{n}$ also figures prominently in the definition of the Catalan numbers:

$$C(n) = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

They too have a rich history with many combinatorial applications (see Stanley [**11**]). Note that $C(n)$ is an integer, that is, $n + 1$ divides $\binom{2n}{n}$.

This paper originated with the naive question: Is there some number k other than 1 such that $n + k$ always divides $\binom{2n}{n}$?

It turns out that the short answer is "no." For each $k \neq 1$ there are infinitely many n with $n + k$ not dividing $\binom{2n}{n}$. However, if $k \geq 2$, $n + k$ "usually" divides $\binom{2n}{n}$ (in a way that we will make precise). On the other hand, we show that if $k \leq 0$, $n + k$ divides $\binom{2n}{n}$ less frequently than not, leaving unresolved the precise nature of this frequency. While not particularly deep, these results appear to be new. The proofs stem from a number-theoretic (as opposed to combinatorial) proof that $C(n)$ is integral.

Along the way we shall meet the notorious problem of Ron Graham (which has a cash prize attached) on whether there are infinitely many numbers n with $\binom{2n}{n}$ relatively prime to 105.

But let us begin at the very beginning, taking nothing for granted.

2. WHY ARE THE BINOMIAL COEFFICIENTS INTEGERS? The binomial coefficient $\binom{m}{k}$, defined as

$$\frac{m!}{k!(m-k)!},$$

is integral. Really? Perhaps it is not so obvious that the denominator divides the numerator. There are numerous proofs of course, for example one can use induction and Pascal's rule:

$$\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}.$$

Or one can argue combinatorially that $\binom{m}{k}$ counts the number of k-element subsets of an m-element set.

Instead of looking at the shortest proof, let's instead consider a more complicated proof, one that allows us to introduce some useful notation and also prove other results.

For a prime p and a positive integer m, let $v_p(m)$ denote the number of factors of p in the prime factorization of m. For example, $v_2(10) = 1$, $v_2(11) = 0$, and $v_2(12) = 2$. This function can be extended to positive rational numbers via $v_p(a/b) = v_p(a) - v_p(b)$. Thus, a/b is integral if and only if $v_p(a/b) \geq 0$ for all primes p.

The power of p in $m!$ is given by

$$v_p(m!) = \sum_{k=1}^{m} v_p(k) = \sum_{k=1}^{m} \sum_{j=1}^{v_p(k)} 1 = \sum_{j \geq 1} \sum_{\substack{k=1 \\ p^j | k}}^{m} 1 = \sum_{j \geq 1} \left\lfloor \frac{m}{p^j} \right\rfloor,$$

the last step using that $\lfloor m/p^j \rfloor$ is the number of multiples of p^j in $\{1, 2, \ldots, m\}$. This result, sometimes referred to as Legendre's formula, is well known in elementary number theory. We also have the almost trivial inequality

$$\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor \tag{2}$$

for all real numbers x, y. From these two results, it is immediate that $\binom{m}{k}$ is an integer. Indeed, for each prime p we have

$$v_p \left(\binom{m}{k} \right) = \sum_{j \geq 1} \left(\left\lfloor \frac{m}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{m-k}{p^j} \right\rfloor \right) \geq \sum_{j \geq 1} 0 = 0. \tag{3}$$

3. KUMMER'S THEOREM AND CATALAN NUMBERS. There is another consequence of this line of thinking. For a real number x let $\{x\} = x - \lfloor x \rfloor$, the fractional part of x. The inequality (2) can be improved to an equation:

$$\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor = \{x\} + \{y\} - \{x + y\} = \begin{cases} 1, & \text{if } \{x\} + \{y\} \geq 1, \\ 0, & \text{if } \{x\} + \{y\} < 1. \end{cases}$$

So by (3), $v_p(\binom{m}{k})$ is the number of values of j such that $\{k/p^j\} + \{(m-k)/p^j\}$ is at least 1.

Let's write k and $m - k$ in the base p, so that

$$k = a_0 + a_1 p + \cdots, \quad m - k = b_0 + b_1 p + \cdots,$$

where the "digits" a_i, b_i are integers in the range 0 to $p - 1$. For $j \geq 1$,

$$\left\{ \frac{k}{p^j} \right\} = \frac{a_0 + a_1 p + \cdots + a_{j-1} p^{j-1}}{p^j}, \quad \left\{ \frac{m-k}{p^j} \right\} = \frac{b_0 + b_1 p + \cdots + b_{j-1} p^{j-1}}{p^j},$$

and so we see that $\{k/p^j\} + \{(m-k)/p^j\} \geq 1$ if and only if in the addition of k and $m - k$ in the base p there is a carry into place j caused by the earlier digits.

This implies the remarkable result of Kummer from 1852: *For each prime p and integers $0 \leq k \leq m$, $v_p \left(\binom{m}{k} \right)$ is the number of carries in the addition $k + (m - k) = m$ when done in the base p.*

Recall the definition (1) of the Catalan number $C(n)$. Any of the numerous combinatorial applications of the Catalan numbers lead to a proof that $C(n)$ is an integer, being the solution to a counting problem. From a number-theoretic perspective, one can ask for a direct proof of the integrality of $C(n)$. This is not difficult, perhaps the easiest way to see it is via the identity

$$C(n) = \binom{2n}{n} - \binom{2n}{n-1}.$$

It also follows from Kummer's theorem. Indeed, say p is a prime and $v_p(n+1) = j$. Then the least significant j digits in base p of $n+1$ are all 0, so the least significant j digits in base p of n are all $p-1$. Thus, in the addition $n+n = 2n$ performed in the base p, we have j carries from the least significant j digits, and perhaps some other carries as well. So we have $v_p(\binom{2n}{n}) \geq j$. Since this is true for all primes p, we have $n+1 \mid \binom{2n}{n}$ and so $C(n) = \binom{2n}{n}/(n+1)$ is an integer.

It may seem that the path of this paper is to give difficult proofs of easy theorems! However, we shall see that this proof of the integrality of $C(n)$ "has legs" and can be used to also prove some perhaps surprising new results. But first we take a not-unrelated detour to view the notorious 105 problem.

4. WHEN IS $\binom{2n}{n}$ RELATIVELY PRIME TO 105?
The numbers $n = 1$, 10, and 756 have $\binom{2n}{n}$ relatively prime to 105. Are there infinitely many others? This problem is due to Ron Graham, and according to [2, 4], Graham offers a prize of $1,000 to settle it.

What's the deal with 105? It is $3 \times 5 \times 7$, the product of the first three odd primes. More generally, one can ask for any fixed number m divisible by at least three distinct odd primes, if $\binom{2n}{n}$ is relatively prime to m for infinitely many n. In the case when $m = pq$, the product of just two odd primes, we do know that $\binom{2n}{n}$ is relatively prime to m for infinitely many n, a result of Erdős, Graham, Ruzsa, and Straus, see [3].

These problems and results stem from the point of view taken in Kummer's theorem, discussed in the previous section. We can use Kummer's theorem to show that $\binom{2n}{n}$ is usually divisible by all small primes p. This is obvious for $p = 2$ since in the base 2, the number $2n$ has one more digit than n so there is at least one carry in the addition $n + n$; that is, $\binom{2n}{n}$ is always even.

For an odd prime p, let

$$R_p = \left\{ 0, 1, \ldots, \frac{1}{2}(p-1) \right\}, \quad r_p = \#R_p = \frac{1}{2}(p+1), \quad \theta_p = \frac{\log r_p}{\log p}.$$

We see that p does not divide $\binom{2n}{n}$ precisely when all of the base-p digits of n come from R_p. We show this is an unusual event.

Lemma 1. *For each odd prime p and all real numbers $x \geq 2$, the number of integers $1 \leq n \leq x$ with $p \nmid \binom{2n}{n}$ is at most px^{θ_p}.*

Proof. If $p \nmid \binom{2n}{n}$, then by Kummer's theorem, every base-p digit of n is in R_p. Let $D = \lfloor 1 + \log x / \log p \rfloor$, so that if $1 \leq n \leq x$ is an integer, then n has at most D base-p digits. If we restrict these digits so that they are in R_p, we would have at most

r_p choices in each place. Thus, the number of choices for n is at most r_p^D. It remains to note that

$$r_p^D < pr_p^{\log x / \log p} = px^{\theta_p},$$

so concluding the proof. ∎

For example, there are at most $3x^{\theta_3}$ integers $n \leq x$ with $\binom{2n}{n}$ not divisible by 3. Since the exponent $\theta_3 = \log 2 / \log 3 = 0.6309\ldots$ is smaller than 1, we see that the vast majority of integers n up to x have $\binom{2n}{n}$ divisible by 3.

Here is a heuristic argument for why there are infinitely many n with $\binom{2n}{n}$ relatively prime to 105. The idea is to view Lemma 1 as an assertion about the probability that p does not divide $\binom{2n}{n}$ when n is randomly chosen in $[1, x]$. When x is an integer, this probability is at most $px^{\theta_p - 1}$. The exponents $\theta_p - 1$ for $p = 3, 5, 7$ are greater than -0.37, -0.32, and -0.29, respectively. If these events are *independent*, as would seem only just (why would the base-p expansion of n have anything to do with the base-q expansion when p and q are different primes?), then the probability that $\binom{2n}{n}$ is relatively prime to 105, where $n \leq x$, exceeds $x^{-0.98}$ when x is large. Thus, we expect at least $x^{0.02}$ examples, and this expression tends to infinity when $x \to \infty$, albeit fairly slowly.

It is interesting to note that if one redoes this heuristic for the four primes 3, 5, 7, 11, then it suggests that there are at most finitely many numbers, such as $n = 3160$, where $\binom{2n}{n}$ is relatively prime to 1155. No example larger than 3160 is known, though they have been searched for up to 10^{10^4}, see [7].

5. HOW FREQUENTLY DOES $n + k$ DIVIDE $\binom{2n}{n}$?

We have seen that $n + 1$ divides $\binom{2n}{n}$ for all n. We now ask what happens with $n + k$ when $k \neq 1$. Is there a value of k where $n + k$ divides $\binom{2n}{n}$ for all n or for all sufficiently large n?

We prove the following results which show an important cleavage between the cases when $k \geq 2$ and the cases when $k \leq 0$. However, we first state a universal result.

Theorem 1. *For each integer $k \neq 1$ there are infinitely many positive integers n with $n + k \nmid \binom{2n}{n}$.*

Thus, the case $k = 1$ of Catalan numbers is indeed special. However, the set of numbers n satisfying the condition of Theorem 1 when $k \geq 2$ is rather sparse. To measure how dense or sparse a set S of positive integers is, let $S(x)$ denote the number of members of S in $[1, x]$. Then the "asymptotic density" of S is $\lim_{x \to \infty} S(x)/x$ if this limit exists. In general, the limsup gives the upper asymptotic density of S and the liminf the lower asymptotic density. For example, the set of odd numbers has asymptotic density $\frac{1}{2}$, the set of prime numbers has asymptotic density 0, and the set of numbers which have an even number of decimal digits has upper asymptotic density $10/11$ and lower asymptotic density $1/11$.

Theorem 2. *For each positive integer k, the set of positive integers n with*

$$n + k \mid \binom{2n}{n}$$

has asymptotic density 1.

So it is common for $n + k$ to divide $\binom{2n}{n}$, but what about $n - k$?

Theorem 3. *For each integer $k \geq 0$ the set of integers $n > k$ with $n - k \mid \binom{2n}{n}$ is infinite, but has upper asymptotic density smaller than $\frac{1}{3}$.*

A remark: That there are infinitely many integers n with $n \mid \binom{2n}{n}$ and also infinitely many with $n \nmid \binom{2n}{n}$ follow from Theorems 3.2 and 3.4 in the recent paper of Ulas [**13**].

6. THE PROOFS OF THEOREMS 1, 2, AND 3. There is a quick proof of Theorem 1. First assume that $k \geq 2$. Let p be a prime factor of k and let $n = p^j - k$ where j is large enough so that $n > 0$. In base p, n has at most j digits, with the least significant digit being 0. Hence, there are at most $j - 1$ carries when adding n to n and hence $n + k = p^j \nmid \binom{2n}{n}$. For $k \leq 0$, let $p > 2|k|$ be an odd prime number. For $n = p + |k|$, we have that there are no carries when n is added to itself in the base p, so that $p \nmid \binom{2n}{n}$. But $p = n + k$, so we are done.

The proof of Theorem 2 is a bit more difficult. We begin with a lemma that extends Lemma 1 to prime powers.

Lemma 2. *Let p be an arbitrary prime, let $x \geq p$ be a real number, and let $D = \lfloor 1 + \log x / \log p \rfloor$. The number of integers $1 \leq n \leq x$ with $v_p\left(\binom{2n}{n}\right) \leq D/(5 \log D)$ is at most $3px^{1-1/(5 \log p)}$.*

Proof. The calculation here is similar to the one in probability where you compute the chance that a coin flipped D times lands heads fairly frequently. We consider the number of assignments of D base-p digits where all but at most $B := \lfloor D/(5 \log D) \rfloor$ of them are smaller than $p/2$. Since there are $\lceil p/2 \rceil$ integers in $[0, p/2)$, the number of assignments is at most

$$\sum_{j=0}^{B} \binom{D}{j} (\lceil p/2 \rceil)^{D-j} (p - \lceil p/2 \rceil)^j = \sum_{j=0}^{B} \binom{D}{j} (\lceil p/2 \rceil)^D \left(\frac{p - \lceil p/2 \rceil}{\lceil p/2 \rceil} \right)^j$$

$$\leq (\lceil p/2 \rceil)^D \sum_{j=0}^{B} \binom{D}{j}.$$

A crude estimation using $D \geq 2$ gets us

$$\sum_{j=0}^{B} \binom{D}{j} \leq \sum_{j=0}^{B} D^j < 2D^B,$$

so the number of $n \leq x$ with $v_p\left(\binom{2n}{n}\right) \leq B$ is at most $2(\lceil p/2 \rceil)^D D^B$. Now

$$2D^B = 2e^{B \log D} \leq 2e^{D/5} \leq 2e^{1/5} x^{1/(5 \log p)},$$

$\lceil p/2 \rceil \leq \frac{2}{3} p$, $\left(\frac{2}{3}\right)^D \leq x^{-\log(3/2)/\log p}$, and $p^D \leq px$. We conclude that the number of $n \leq x$ with $v_p\left(\binom{2n}{n}\right) \leq B$ is at most

$$2\left(\lceil p/2 \rceil\right)^D D^B \leq 2D^B (\tfrac{2}{3} p)^D \leq 2e^{1/5} px^{1-\log(3/2)/\log p + 1/(5 \log p)}.$$

Since $2e^{1/5} < 3$ and $\log(3/2) > 2/5$, the lemma follows at once. ∎

We are now ready to prove Theorem 2. Fix a value of $k \geq 1$. First we claim that for $p \geq 2k$,

$$v_p\left(\binom{2n}{n}\right) \geq v_p(n+k). \tag{4}$$

Indeed, if $v_p(n+k) = j > 0$, then the j least significant digits of $n + k$ in base p are 0, so the j least significant digits of n are at least $p - k \geq p/2$. By Kummer's theorem, (4) holds. (Note that this argument is essentially a reprise of the proof given in Section 3 that Catalan numbers are integers.)

It remains to show that for "most" integers n, (4) holds for all primes $p < 2k$. Let $x \geq 2k + 1$ be a real number and assume that we are considering values of $n \leq x$. With D as in Lemma 2, we consider two cases: those n with $v_p(n+k) \leq D/(5 \log D)$ and those n with $v_p(n+k) > D/(5 \log D)$. In the first case, if (4) fails, we would have $v_p(\binom{2n}{n}) < D/(5 \log D)$, so by Lemma 2, the number of n being considered is at most $3px^{1-1/(5 \log p)}$. Summing this expression for primes $p < 2k$ gives a quantity smaller than $6k^2 x^{1-1/(5 \log(2k))}$. Divided by x, this expression tends to 0 as $x \to \infty$, so we are left with the second case: those n with $v_p(n+k) > D/(5 \log D)$. We shall show in this case that there are very few values of $n \leq x$ to consider, regardless if (4) holds. In fact, the number of choices for n is at most $(x+k)/p^{\lceil D/(5 \log D)\rceil} < 2x/p^{D/(5 \log D)}$. Since $p^D > p^{\log x / \log p} = x$, we thus have that the number of choices for n in the second case corresponding to the prime $p < 2k$ is at most $2x^{1-1/(5 \log D)}$. We have $\log D \leq \log(1 + \log x / \log 2) < 1 + \log \log x$ (using $\log(a+b) < a/b + \log b$ when $0 < a < b$), so that if we sum for $p < 2k$, we get that the number of choices for n is at most $2kx^{1-1/(5+5 \log \log x)}$. When divided by x, this too goes to 0 as $x \to \infty$, which completes the proof of Theorem 2.

It is not difficult to amend the proof to show that for each fixed positive integer k, the set of integers n with

$$(n+1)(n+2)\cdots(n+k) \,\Big|\, \binom{2n}{n} \tag{5}$$

has asymptotic density 1. In addition, the proof allows for k to tend to infinity, provided it does not do so too quickly in comparison with x. The result (5) might be compared with Harborth [6] where it is shown that for any fixed positive integer k, "almost all" entries $\binom{m}{j}$ in Pascal's triangle are divisible by $m(m-1)\cdots(m-k+1)$. Also see http://oeis.org, sequences A065344–9.

We now proceed to the proof of Theorem 3, starting with the second assertion. Suppose that $k \geq 0$, $n > 2k^2$, and $m = n - k$ has a prime factor $p > \sqrt{2n}$. Then $p > 2k$ and writing $m = cp$, we have

$$c < \frac{m}{\sqrt{2n}} \leq \frac{n}{\sqrt{2n}} = \frac{1}{2}\sqrt{2n} < \frac{1}{2}p.$$

We see that both base-p digits of $n = cp + k$, namely c and k, are smaller than $\frac{1}{2}p$, and so there are no carries when adding n to itself in base p. By Kummer's theorem, we have $p \nmid \binom{2n}{n}$, so that $n - k \nmid \binom{2n}{n}$.

Suppose that $x > 2k^4$. To show the second assertion in the theorem, it will suffice to show that when x is sufficiently large, at least $\frac{2}{3}x$ integers $n \in (2k^2, x]$ have $m = n - k$ divisible by a prime $p > \sqrt{2x}$. For each prime p satisfying this inequality, we count

numbers n in $(2k^2, x]$ with $p \mid n - k$, and this is at least $\lfloor (x - 2k^2)/p \rfloor > x/p - 2$. No choice of n corresponds to two different values of p, since their product would be too large to have $n \le x$. Thus, the number of choices for n is at least

$$\sum_{\sqrt{2x} < p \le x} \left(\frac{x}{p} - 2 \right) = x \sum_{\sqrt{2x} < p \le x} \frac{1}{p} - 2\pi(x),$$

where $\pi(x)$ denotes the number of primes in $[1, x]$. Euler proved long ago in 1737 that the sum of the reciprocals of the primes diverges to infinity like the double-log function, and using a finer estimation (such as the theorem of Mertens from 1874), we have

$$\sum_{\sqrt{2x} < p \le x} \frac{1}{p} = \log \log x - \log \log(\sqrt{2x}) + E(x),$$

where $E(x) \to 0$ as $x \to \infty$, see [**8, 9**]. The difference of double logs simplifies to

$$\log 2 + \log \left(\frac{\log x}{\log(2x)} \right),$$

which tends to $\log 2$ as $x \to \infty$. As mentioned before, the set of primes has asymptotic density 0, in fact $\pi(x)/x$ goes to 0 like $1/\log x$ by Chebyshev's estimates or the prime number theorem. Putting these thoughts together, it follows that for each $\varepsilon > 0$ and x sufficiently large, there are more than $(\log 2 - \varepsilon)x$ values of $n \in (2k^2, x]$ with $n - k$ divisible by a prime $p > \sqrt{2x}$. We have seen that for each such number n, we have $n - k \nmid \binom{2n}{n}$. Since $\log 2 > 0.6931 > \frac{2}{3}$, the second part of Theorem 3 follows.

To complete the proof, we wish to show there are infinitely many values of n where $n - k \mid \binom{2n}{n}$. We leave the few details for the reader, but using Kummer's theorem, we have that if $n = pq + k$ where p, q are primes with $k < p$ and $\frac{3}{2}p < q < 2p$, then $n - k \mid \binom{2n}{n}$. The number of such numbers $n \le x$ is greater than a positive constant times $x/(\log x)^2$.

7. THE "GOVERNOR SET." For each integer k, let

$$D_k = \left\{ n \ : \ n + k \ \Big| \ \binom{2n}{n} \right\}.$$

We call D_0 the *governor set* for a reason that will soon be clear.

Say two sets A, B of positive integers are *asymptotically equivalent* if the symmetric difference $(A \cup B) \setminus (A \cap B)$ has asymptotic density 0. In this case, we write $A \simeq B$. For example, if A is the set of all positive integers exceeding a googol (10^{100}) and B is the set of all composite positive integers, then $A \simeq B$. In particular, any two sets of asymptotic density 1 are asymptotically equivalent, as are any two sets of asymptotic density 0.

If A is a set of positive integers and n is a positive integer, we let

$$A + n := \{a + n : a \in A\}.$$

Theorem 4. *For each positive integer k we have $D_0 + k \simeq D_{-k}$.*

We sketch the proof. Suppose that p is a prime with $p \mid n$ and $p > 2k$. Then, as in the proof of Theorem 2, $v_p(\binom{2n}{n}) = v_p(\binom{2(n+k)}{n+k})$. And for primes p at most $2k$, again following the argument for Theorem 2, for most numbers n, the power of p in both $\binom{2n}{n}$ and in $\binom{2(n+k)}{n+k}$ is higher than the power of p in n. Thus, most of the time, the condition $n \in D_0$ (that is, $n \mid \binom{2n}{n}$) is equivalent to the condition $n + k \in D_{-k}$ (that is, $n \mid \binom{2(n+k)}{n+k}$).

It is not clear if the governor set D_0 has positive lower asymptotic density, though I conjecture this the case. In Theorem 3 we essentially learned that the upper asymptotic density is at most $1 - \log 2$, and perhaps some improvement can be made. It also would be highly interesting to investigate the problem numerically. The numbers n which divide $\binom{2n}{n}$ are 1, 2, 6, 15, 20, 28, 42, 45, 66, 77, 88, 91, ... What are these numbers trying to tell us? A census to higher levels may be illuminating. Note that D_0 may also be described as the set of n with $n \mid C(n)$, and in this guise there is some information to be found at http://oeis.org, sequence A014847.

To cement the role of the governor set D_0, let

$$D_k^{(2)} = \left\{ n \ : \ (n+k)^2 \ \Big| \ \binom{2n}{n} \right\}.$$

Then for each positive integer k we have $D_k^{(2)} + k \simeq D_0$. The proof is similar to that of Theorem 4. So you'd like to know how often the nth Catalan number $C(n)$ is divisible by $n + 1$? You are again led back to the governor set D_0. See too [1] and http://oeis.org, sequence A002503. (Notice that $n + 1 \mid C(n)$ if and only if $n + 1 \mid \binom{2n+1}{n}$. This latter condition is considered in [13], where in Question 2.8(a), the author asks about the distribution of such n. Our Theorem 3 in the case $k = 0$ is thus relevant to this query.)

You may have noticed that any D_k with $k \leq 0$ could have played the special role of the governor set, but surely D_0 is the most pleasing.

8. OTHER BINOMIAL COEFFICIENTS.
One might wonder what the fuss is about the middle entry $\binom{2n}{n}$ in the $2n$th row of Pascal's triangle. What about odd-numbered rows, where there are the twin peaks $\binom{2n+1}{n} = \binom{2n+1}{n+1}$? It is possible to prove corresponding results on divisibility here. For example, for $k \geq 2$, $(n+2)(n+3)\cdots(n+k)$ usually divides $\binom{2n+1}{n}$. If you are interested, you should try to prove this.

Looking at just the case $k = 2$, we have a near miss for $n + 2$ always dividing $\binom{2n+1}{n}$. In fact, unless $n + 2$ is a power of 2, it divides, and even in this case, it divides $2\binom{2n+1}{n}$. Since there is a tie for the maximum entry in this row of Pascal's triangle, it makes sense to include both of them, and as mentioned we always have

$$n + 2 \ \Big| \ 2\binom{2n+1}{n}.$$

Thus, we might wonder, as with Catalan numbers, if the integer

$$\frac{2}{n+2}\binom{2n+1}{n} \tag{6}$$

has combinatorial significance. Indeed it does. We know that the Catalan number $C(n)$ counts the number of paths from $(0, 0)$ to (n, n) that do not cross below the line $y = x$, and where each step of the path is one unit to the right or one unit up. The number in

(6) is similar, but now we are counting paths from $(0, 0)$ to $(n, n + 1)$, a so-called ballot number.

In [**12**], Sun has the following generalization of the fact that $n + 1$ always divides $\binom{2n}{n}$. If k, ℓ are positive integers and every prime dividing k also divides ℓ, then

$$\ell n + 1 \,\Big|\, \binom{(\ell + k)n}{kn} \text{ for every positive integer } n. \tag{7}$$

The case $k = \ell = 1$ is the situation with Catalan numbers. Sun asks if the condition that every prime factor of k divides ℓ is necessary for (7) to hold.

For other problems and results concerning divisibility properties of binomial coefficients, the reader is referred to [**3, 5, 10**].

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