

REMARKS ON THE MIDDLE BINOMIAL COEFFICIENT

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Abstract

We show that $\binom{m+k}{k} \mid \binom{2m}{m}$ for all $k \leq \exp(.8\sqrt{\log m})$ on a set of numbers m of asymptotic density 1. We also show that $(m+1)(m+2)\cdots(m+k)$ divides $\binom{2m}{m}$ on a set of asymptotic density 1 for k as large as $.7 \log m$.

1. Introduction

In [4] I proved some elementary results about the middle binomial coefficient $\binom{2m}{m}$. In particular, Theorem 2 of that paper shows that for each fixed positive integer k , we have $m+k \mid \binom{2m}{m}$ for a set of integers m of asymptotic density 1. Following the proof it is left as an exercise to show that the product $(m+1)(m+2)\cdots(m+k)$ divides $\binom{2m}{m}$ on a set of asymptotic density 1. We prove the following strengthening.

Theorem 1. *For any fixed $\eta < 1/\log 4 = .721\dots$, we have for a set of integers m of asymptotic density 1 that*

$$\frac{(m+k)!}{m!} \mid \binom{2m}{m} \tag{1.1}$$

for all positive integers $k \leq \eta \log m$.

Replacing the product with $\binom{m+k}{k}$, we prove the following theorem.

Theorem 2. *For m in a set of asymptotic density 1, we have*

$$\binom{m+k}{k} \mid \binom{2m}{m} \tag{1.2}$$

for all positive integers $k \leq \exp(.8\sqrt{\log m})$.

These theorems may be of interest in the context of recent work towards settling a problem of Erdős using AI.¹ One might compare this note with the write-up of Sothanaphan [6].

¹See T. Tao, <https://mathstodon.xyz/@tao/115855840223258103>.

The paper [4] spawned some other work as well. Since $m + 1$ always divides $\binom{2m}{m}$ and $m + k \mid \binom{2m}{m}$ almost always when $k > 1$, what about $m \mid \binom{2m}{m}$? It was shown that the upper density of this set is at most $1 - \log 2$ and conjectured that the density exists and is positive. Sanna [6] showed that the upper density of the set is smaller than $1/4$, while Ford and Konyagin [3] proved the conjecture, showing the density is slightly larger than $1/11$. In fact, they showed that for each fixed positive integer ℓ , the density of the set of m with $m^\ell \mid \binom{2m}{m}$ exists and is positive.

2. Proof of Theorem 1

We let p denote a prime variable, and we let v_p be the function which returns the exponent on p in the prime factorization of its argument.

Lemma 1. *Let*

$$\alpha_p(m) = \frac{v_p\left(\binom{2m}{m}\right)}{\log m / \log p}.$$

For a set of integers m of asymptotic density 1 we have $\alpha_2(m) = 1/2 + o(1)$, $\alpha_3(m) \geq 34/81 + o(1)$, and $\alpha_p(m) \geq .39$ uniformly for all $3 < p < 2 \log m$.

Proof. Note that m has $\lceil \log m / \log p \rceil + 1$ base- p digits and we expect roughly half of these to be at least $p/2$. More precisely, for $p = 2$, we expect half to be at least $p/2$ and for p odd we expect $(p - 1)/2p$ of them to be at least $p/2$. The probability of getting at most $.4N$ heads when flipping a fair coin N times is

$$2^{-N} \sum_{k \leq .4N} \binom{N}{k} = e^{(-.4 \log .4 - .6 \log .6 + o(1))N} 2^{-N} < e^{-.02N}$$

for N large. For $3 < p < 2 \log x$ a calculation using this probability shows that the number of $m \leq x$ with fewer than $.39 \log m / \log p$ base- p digits at least $p/2$ is $O(x^{1-c/\log p})$. Here c is a small positive constant. Summing this for $p < 2 \log x$ we get an expression that is $o(x)$ as $x \rightarrow \infty$. We can add to this exceptional set those m with $\alpha_2(m) \leq 1/2 - \epsilon$ for any fixed positive ϵ , with c now depending on ϵ . For $p = 3$ we consider the base-27 expansion of m , finding that the average number of base-3 carries in doubling a base-27 digit is $34/27$, so our assertion about $\alpha_3(m)$ follows as well. □

If $p > 2k$, then $v_p((m + k)!/m!) = \max\{v_p(m + i) : 1 \leq i \leq k\}$. As in [4], if this max is j , occurring at $m + i_0$, then the j least significant base- p digits of $m + i_0$ are 0, so the j least significant base- p digits of m are $p - i_0 \geq p - k > p/2$, and so $v_p\left(\binom{2m}{m}\right) \geq j = v_p((m + k)!/m!)$. So assume that $p \leq 2k$.

Lemma 2. *For a set of integers m of asymptotic density 1 we have*

$$\max\{v_p(m+i) : 1 \leq i \leq k\} \leq 3 \frac{\log k}{\log p}$$

for all k with $\frac{1}{2} \log m < k < \log m$ and for all $p \leq 2k$.

Proof. We may assume that $x/\log x < m \leq x$. If $1 \leq i \leq k$, the number of $m \leq x$ with $v_p(m+i) > 3 \log k / \log p$ is at most x/k^3 . Summing this for $i \leq k$ and $p \leq 2k$, the count is $\ll x/k = o(x)$. Thus, but for these exceptional values of m we have the inequality in the lemma. \square

Lemma 3. *For all integers $m, k > 0$ and primes p , we have*

$$v_p \left(\frac{(m+k)!}{m!} \right) \leq v_p(k!) + \max\{v_p(m+i) : 1 \leq i \leq k\}.$$

Proof. We may assume that the max in the lemma is positive, say it first occurs at $m+i_0$. If $p^j \mid m+i_0$, then the number of multiples of p^j in $\{m+1, \dots, m+k\} \setminus \{m+i_0\}$ is at most $\lceil k/p^j \rceil - 1 \leq \lfloor k/p^j \rfloor$. Summing on j we have

$$v_p \left(\frac{(m+k)!}{m!} \right) \leq v_p(m+i_0) + \sum_{p^j \mid m+i_0} \lfloor k/p^j \rfloor \leq v_p(m+i_0) + v_p(k!),$$

which was to be proved. \square

As a corollary, we have for all integers $m, k > 0$ and primes p ,

$$v_p \left(\binom{m+k}{k} \right) \leq \max\{v_p(m+i) : 1 \leq i \leq k\}. \quad (2.1)$$

Using Lemmas 2, 3 we have on a set of integers m of density 1 and for $p \leq 2k$, $\frac{1}{2} \log m < k < \log m$ that

$$v_p((m+k)!/m!) \leq v_p(k!) + \max\{v_p(m+i) : 1 \leq i \leq k\} < \frac{k}{p-1} + 3 \frac{\log k}{\log p}.$$

For a fixed $\epsilon > 0$, let $\beta_2 = 1/2 - \epsilon$, $\beta_3 = 34/81 - \epsilon$, and $\beta_p = .39$ for $p > 3$. It remains for us to find, in light of Lemma 1, how large we may take k so that

$$\frac{k}{p-1} + 3 \frac{\log k}{\log p} \leq \beta_p \frac{\log m}{\log p}$$

holds for all $p \leq 2k$. The most difficult value of p to accommodate is $p = 2$ and we find that the inequality holds for k as large as $(1/\log 4 - 2\epsilon) \log m$.

Remark. It should be possible to show in Lemma 1 that $\alpha_p \sim 1/2$ for all $p \leq 2 \log x$. A result in this direction is the theorem at the bottom of page 89 of [2]. The constant $1/\log 4$ is optimal in that if k is slightly larger, the set of m where the divisibility holds does not have density 1. In fact, a somewhat larger constant times $\log m$ for k would eliminate all examples, cf. [1].

3. Proof of Theorem 2

Let x be large, $D = D_p = 1 + \lfloor \log x / \log p \rfloor$, and $K = \lfloor \exp(.8\sqrt{\log x}) \rfloor$.

Lemma 4. *The number of integers $m \leq x$ with $v_p(\binom{2m}{m}) \leq D/\log D$ for some prime $p \leq 2K$ is $o(x)$.*

Proof. We follow the proof of Lemma 2 in [4], with some improvements. Let $B = D/\log D$ and fix a prime $p \leq 2K$. The number of $m \leq x$ with fewer than B base- p digits at least $p/2$ is smaller than

$$\lceil p/2 \rceil^D \sum_{j < B} \binom{D}{j}.$$

Since B is small compared to D , the sum here is $\ll \binom{D}{\lfloor B \rfloor}$. A short calculation with Stirling's formula shows that this expression is at most $\exp(O(D \log \log D / \log D))$, so $\binom{D}{\lfloor B \rfloor} \leq x^{.01/\log p}$ for x larger than an absolute constant. Replacing $\lfloor p/2 \rfloor$ with $p/2$ creates an error smaller than $x^{.01/\log p}$, so our count is then at most

$$p^D 2^{-D} x^{.02/\log p} \leq x^{1-(2/3)/\log p}$$

for x large. Summing this for $p \leq 2K$ there are at most $Kx^{1-(2/3)/\log(2K)}$ choices of $m \leq x$ with $v_p(\binom{2m}{m}) \leq B$. Since this bound is $o(x)$ as $x \rightarrow \infty$, the lemma is proved. \square

Lemma 5. *The number of $m \leq x$ such that for some $p < 2K$, $\max\{v_p(m+i) : 1 \leq i \leq K\} > D/\log D$ is $o(x)$ as $x \rightarrow \infty$.*

Proof. The count in question is at most

$$\sum_{p < 2K} \sum_{i \leq K} x/p^{D/\log D} \leq K \sum_{p < 2K} x/p^{D/\log D} < K \sum_{p < 2K} x^{1-1/\log D},$$

since $p^D > x$. As in [4], the summand here is less than $Kx^{1-1/(1+\log \log x)}$, so the count is at most $K^2 x^{1-1/(1+\log \log x)}$, which by our choice of K is $o(x)$. \square

To prove the theorem we need to show that for most integers m

$$v_p \left(\binom{m+k}{k} \right) \leq v_p \left(\binom{2m}{m} \right) \quad (3.1)$$

for all primes p and for all $k \leq K$. First assume that $p > 2k$. Then $v_p \left(\binom{m+k}{k} \right) = v_p((m+k)!/m!)$, so as before (3.1) holds. Thus, it suffices to consider the case that $p \leq 2k$. From Lemma 4 we may assume that $v_p \left(\binom{2m}{m} \right) > D/\log D$ for all $p \leq 2k$.

Using (2.1) and Lemma 5, we may assume that $v_p\left(\binom{m+k}{k}\right) \leq D/\log D$, so the desired inequality follows. This completes the proof of Theorem 2.

Remark. The constant .8 in Theorem 2 may be replaced with any number smaller than $\sqrt{\log 2}$, but beyond that we do not know of any improvements. But at least one can see that if (1.2) holds for asymptotically all m , then for any fixed $\epsilon > 0$ we have $k \ll m^\epsilon$. To see this, take a large prime p and consider integers $m = p^d + c_{d-1}p^{d-1} + \cdots + c_0$, where each $c_i < p/2$. Then $p \nmid \binom{2m}{m}$. Now take $k = p$ so that $p \mid (m+k)!/m!$. The number of choices for m is greater than $p^d/2^d$, and each $m \leq 2p^d$. The proportion of such numbers $m \leq 2p^d$ is at most 2^{1-d} .

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References

- [1] P. Erdős, *Aufgabe 557*, *Elem. Math.* **23** (1968), 111–113.
- [2] P. Erdős, R. L. Graham, I. Ruzsa, and E. G. Straus, On the prime factors of $\binom{2n}{n}$, Collection of articles in honor of Derrick Henry Lehmer on the occasion of his seventieth birthday, *Math. Comp.* **29** (1975), 83–92.
- [3] K. Ford and S. Konyagin, Divisibility of the central binomial coefficient $\binom{2n}{n}$, *Trans. Amer. Math. Soc.* **374** (2021), 923–953.
- [4] C. Pomerance, Divisors of the middle binomial coefficient, *Amer. Math. Monthly* **122** (2015), 636–644.
- [5] C. Sanna, Central binomial coefficients divisible by or coprime to their indices, *Int. J. Number Theory* **14** (2018), 1135–1141.
- [6] N. Sothanaphan, Resolution of Erdős Problem #728: a writeup of Aristotle’s Lean proof, arXiv:2601.07421v4.