Sums and products

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Based on joint work with
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Let's begin with products. Take the $N \times N$ multiplication table. It has N^2 entries. It is a symmetric matrix, so most entries appear at least twice. How many distinct entries does it have?

Let $M(N)$ be the number of distinct entries in the $N \times N$ multiplication table.

$\times$	1	2	3	4	5
1	**1**	**2**	**3**	**4**	**5**
2	2	4	**6**	**8**	**10**
3	3	6	**9**	**12**	**15**
4	4	8	12	**16**	**20**
5	5	10	15	20	**25**

So, $M(5) = 14$.

$\times$	1	2	3	4	5	6	7	8	9	10
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	2	4	6	8	10	**12**	**14**	**16**	**18**	**20**
3	3	6	9	12	**15**	18	**21**	**24**	**27**	**30**
4	4	8	12	16	20	24	**28**	**32**	**36**	**40**
5	5	10	15	20	**25**	30	**35**	40	**45**	**50**
6	6	12	18	24	30	36	**42**	**48**	**54**	**60**
7	7	14	21	28	35	42	**49**	**56**	**63**	**70**
8	8	16	24	32	40	48	56	**64**	**72**	**80**
9	9	18	27	36	45	54	63	72	**81**	**90**
10	10	20	30	40	50	60	70	80	90	**100**

So, $M(10) = 42$.

What would you conjecture about $M(N)$ asymptotically?

Maybe

$$\lim_{N\to\infty} \frac{M(N)}{N^2} = \frac{1}{3}?$$

Maybe

$$\lim_{N\to\infty} \frac{M(N)}{N^2} = c > 0?$$

Maybe

$$\lim_{N\to\infty} \frac{M(N)}{N^2} = 0?$$

Here are some values of $M(N)/N^2$:

N	$M(N)/N^2$
5	0.5600
10	0.4200
20	0.3800
40	0.3231
80	0.3030
160	0.2802
320	0.2671
640	0.2538
1000	0.2481
2000	0.2399
8000	0.2267
16000	0.2215
32000	0.2166

(Calculations by T. D. Noe as reported in the OEIS and by P. Kurlberg.)

Do we have $M(N)$ of the shape N^{2-c_1}?

N	$M(N)/N^2$	c_1
5	0.5600	.3603
10	0.4200	.3768
20	0.3800	.3230
40	0.3231	.3063
80	0.3030	.2725
160	0.2802	.2507
320	0.2671	.2289
640	0.2538	.2122
1000	0.2481	.2018
2000	0.2399	.1878
8000	0.2267	.1651
16000	0.2215	.1557
32000	0.2166	.1475

How about $M(N)$ of the shape $N^2/(\log N)^{c_2}$?

N	$M(N)/N^2$	c_1	c_2
5	0.5600	.3603	1.2184
10	0.4200	.3768	1.0401
20	0.3800	.3230	.8819
40	0.3231	.3063	.8655
80	0.3030	.2725	.8081
160	0.2802	.2507	.7832
320	0.2671	.2289	.7533
640	0.2538	.2122	.7349
1000	0.2481	.2018	.7213
2000	0.2399	.1878	.7038
8000	0.2267	.1651	.6759
16000	0.2215	.1557	.6640
32000	0.2166	.1475	.6539

Or how about $M(N)$ of the shape $N^2/\exp((\log N)^{c_3})$?

N	$M(N)/N^2$	c_1	c_2	c_3
5	0.5600	.3603	1.2184	1.1453
10	0.4200	.3768	1.0401	.1704
20	0.3800	.3230	.8819	.0300
40	0.3231	.3063	.8655	.0935
80	0.3030	.2725	.8081	.1200
160	0.2802	.2507	.7832	.1482
320	0.2671	.2289	.7533	.1585
640	0.2538	.2122	.7349	.1692
1000	0.2481	.2018	.7213	.1718
2000	0.2399	.1878	.7038	.1755
8000	0.2267	.1651	.6759	.1798
16000	0.2215	.1557	.6640	.1808
32000	0.2166	.1475	.6539	.1817

Paul Erdős studied this problem in two papers, one in 1955, the other in 1960.

Paul Erdős, 1913–1996

In 1955, Erdős proved (in Hebrew) that $M(N)/N^2 \to 0$ as $N \to \infty$ and indicated that it was likely that $M(N)$ is of the shape $N^2/(\log N)^c$.

In 1960, at the prodding of Linnik and Vinogradov, Erdős identified (in Russian) the value of "c". Let

$$c = 1 - \frac{1 + \log\log 2}{\log 2} = 0.08607\ldots.$$

Then $M(N^2) = N^2/(\log N)^{c+o(1)}$ as $N \to \infty$.

In work of Tenenbaum progress was made (in French) in nailing down the "$o(1)$".

In 2008, Ford showed (in English) that $M(N)$ is of order of magnitude

$$\frac{N^2}{(\log N)^c(\log\log N)^{3/2}}.$$

No matter the language, we still don't know an asymptotic estimate for $M(N)$, despite this just being about the multiplication table!

So how can the fact that $M(N)$ is small compared to N^2 be explained?

It all comes down to the function $\Omega(n)$, the total number of prime factors of n, counted with multiplicity. For example,

$$\Omega(8) = 3,\ \Omega(9) = 2,\ \Omega(10) = 2,\ \Omega(11) = 1,\ \Omega(12) = 3.$$

Some higher values: $\Omega(1024) = 10$, $\Omega(1009) = 1$, and $\Omega(2^{17} - 1) = 1,\ \Omega(2^{17}) = 17$.

But what is $\Omega(n)$ *usually*? That is, can $\Omega(n)$ be approximately predicted from the size of n if we throw out thin sets like primes and powers of 2?

Indeed it can.

In 1917, Hardy and Ramanujan proved that the normal order of $\Omega(n)$ is $\log\log n$. That is, for each $\epsilon > 0$, the set of integers n with

$$|\Omega(n) - \log\log n| < \epsilon \log\log n$$

has asymptotic density 1.

So, this explains the multiplication table. Most products $n_1 n_2$ have both $n_1 > N^{1/2}$ and $n_2 > N^{1/2}$, and most of these have $\Omega(n_1)$ and $\Omega(n_2)$ fairly close to $\log\log N$ (note that $\log\log(N^{1/2})$ differs from $\log\log N$ by less than 1). So most of the products formed have about $2\log\log N$ prime factors, which is an unusual value to have for a number below N^2.

G. H. Hardy

S. Ramanujan

So, $\log\log N$ for integers below N is the center of the distribution. To quantify $M(N)$ one needs to know about estimates for the tail, and that's where the constant c arises.

I should take a small diversion from our progress here and mention one of the most beautfiful theorems in number theory, the Erdős–Kac theorem. It says that the "standard deviation" for $\Omega(n)$ for integers up to N is $(\log\log N)^{1/2}$ and that the distribution is Gaussian. Namely, for each real number u, the set

$$\{n : \Omega(n) \le \log\log n + u(\log\log n)^{1/2}\}$$

has asymptotic density equal to $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{u} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$

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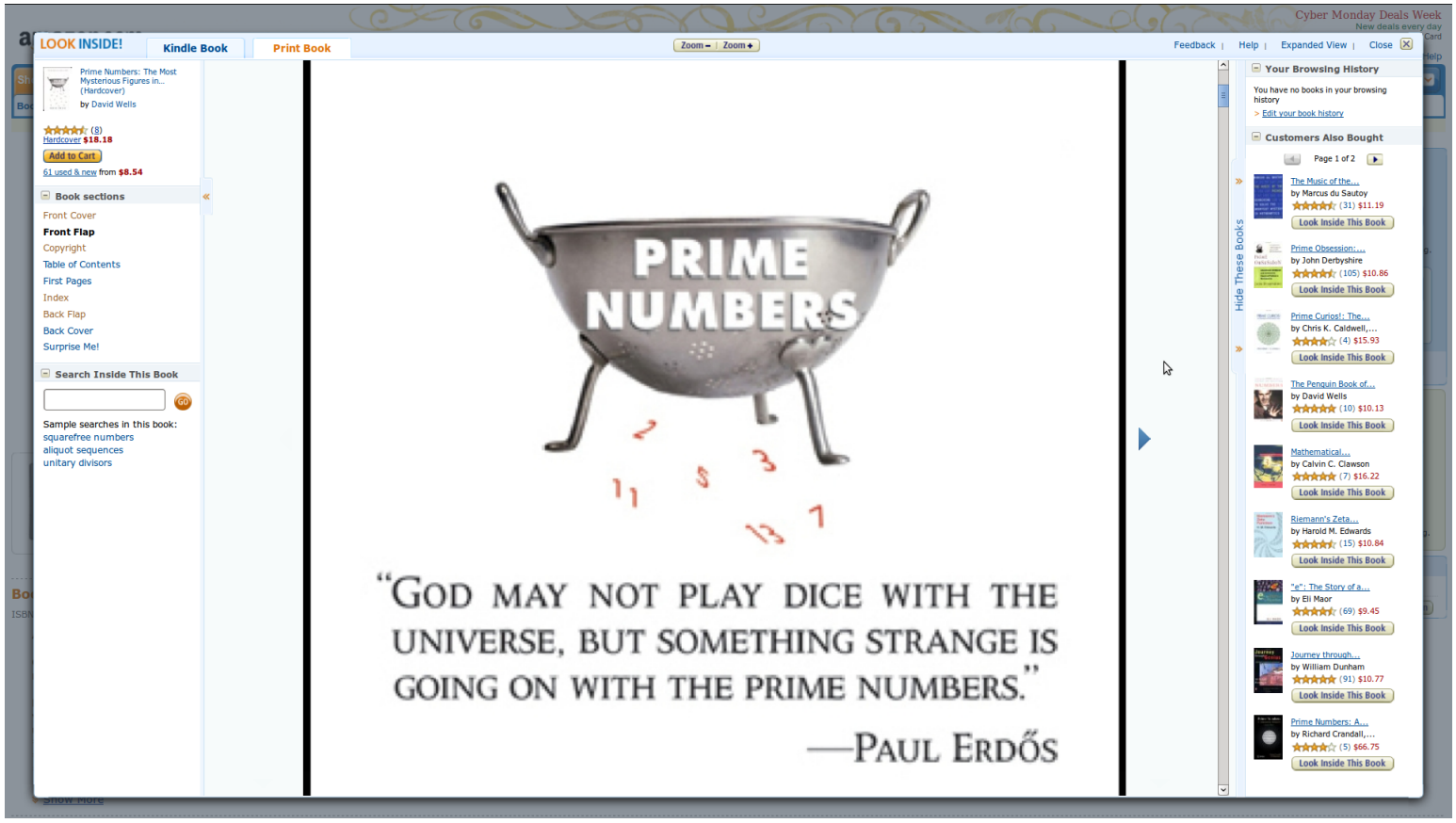
(Note: I made this up, it was a joke . . .)

Prime numbers, the most mysterious figures in math, D. Wells



Keeping with the theme of multiplication, what can be said about sets of positive integers that are *product-free*? This means that for any two members of the set, their product is not in the set. It is as far as you can get from being closed under multiplication.

It is easy to find such sets, for example the set of primes. But how dense can such a set be?

For example, take the integers that are 2 (mod 3). The product of any two of them is 1 (mod 3), so is not in the set. And this set has asymptotic density $\frac{1}{3}$.

Can you do better?

Well, the set of integers that are 2 or 3 (mod 5) is product-free and has density $\frac{2}{5}$.

The set of integers that are 3, 5, or 6 (mod 7) is product-free with density $\frac{3}{7}$.

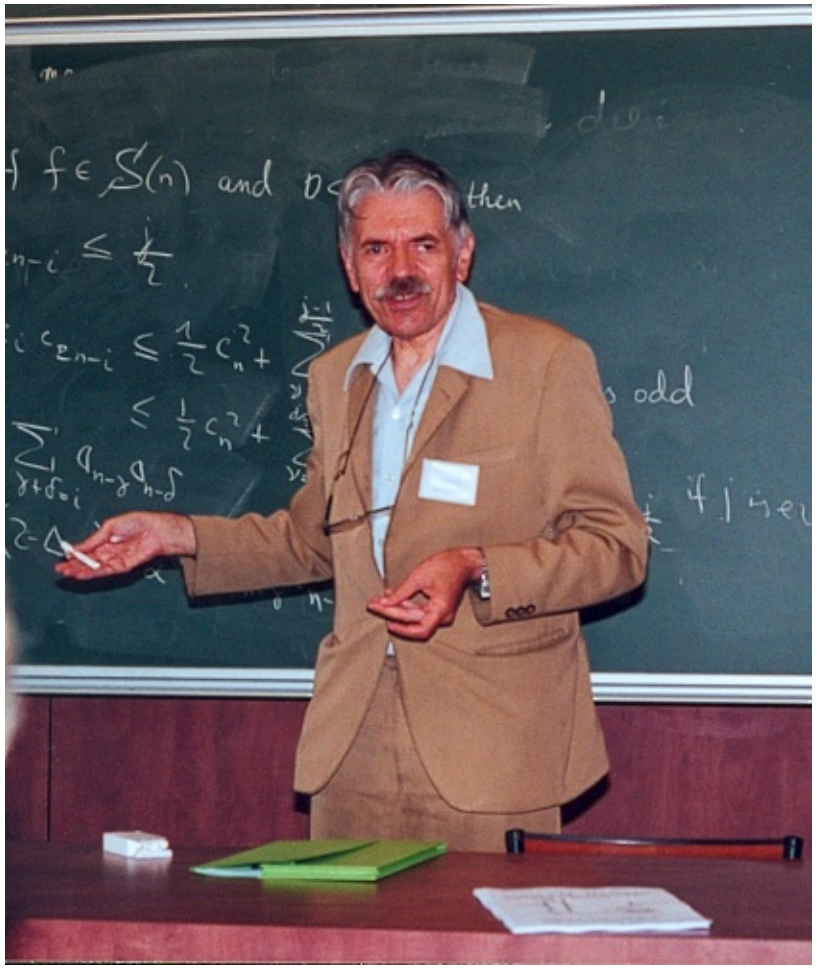
These sets are all described as those integers in certain residue classes modulo some n. Let $D(n)$ denote the maximal possible density of a product-free set modulo n.

It is not hard to prove that $\liminf_{n\to\infty} D(n) = \frac{1}{2}$.

Do we have $D(n) < \frac{1}{2}$ for all n?

P, Schinzel (2011): *We have* $D(n) < \frac{1}{2}$ *for all* n *except possibly those* n *divisible by the square of a number with at least* 6 *distinct prime factors. Further, the asymptotic density of those* n *divisible by such a square is about* 1.56×10^{-8}.

Moscow Journal of Combinatorics and Number Theory,
1 (2011), 52–66.



Andrzej Schinzel

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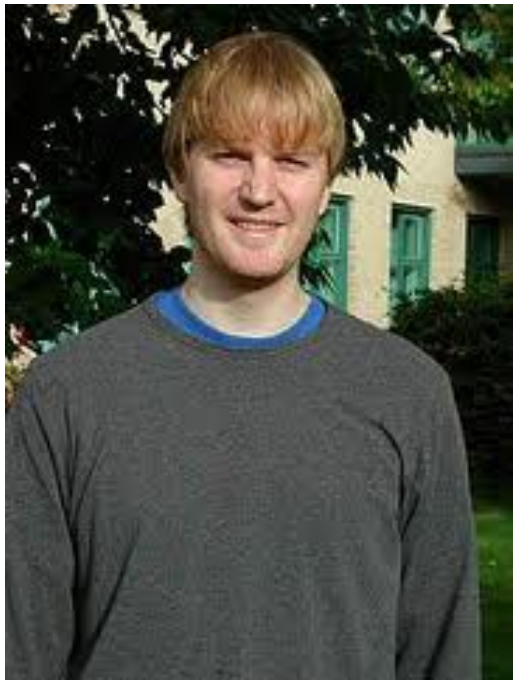
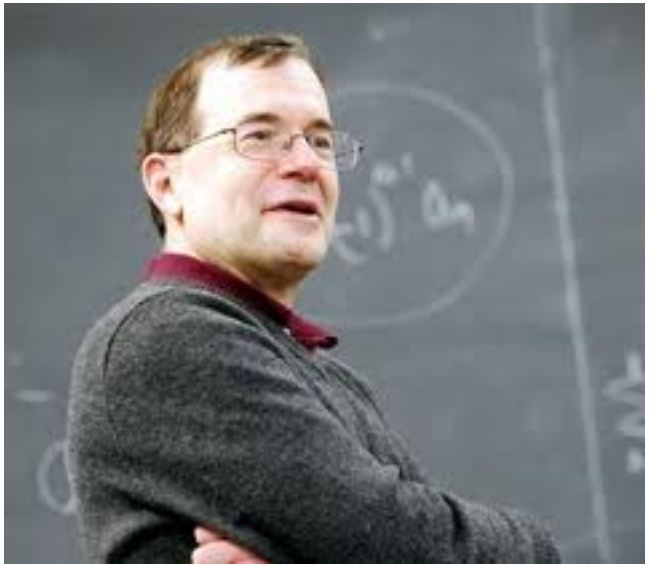
Well, no.

Kurlberg, Lagarias, P (2011): *There are infinitely many values of n with $D(n)$ arbitrarily close to 1. In particular, there are infinitely many values of n where all of the pairwise products of a subset of 99% of the residues $\pmod n$ all fall into the remaining 1% of the residue classes.*

Acta Arithmetica, to appear in a special issue in honor of Andrzej Schinzel's 75th birthday.



Pär Kurlberg



Jeffrey C. Lagarias

Let's be more modest, just show me one n where $D(n) \geq \frac{1}{2}$.

It's not so easy!

Here's a number. Take the first 10,000,000 primes. For those primes below 1,000,000, take their 14th power, and for those that are larger, take their square, and then multiply these powers together to form N. Then $D(N) > 0.5003$. Further, $N \approx 10^{1.61 \times 10^8}$.

Can you find an example with fewer than 100,000,000 decimal digits?

What is behind this construction and proof?

It is actually very similar to the proof of the multiplication table theorem.

Suppose n is a high power of the product of all of the primes up to x, say the exponent is $\lfloor \log x \rfloor$. Then consider all residues $r \pmod n$ with

$$\frac{2}{3} \log \log x < \Omega(\gcd(r, n)) < \frac{4}{3} \log \log x.$$

Then these residues $r \pmod n$ form a product-free set, and in fact most residues (mod n) satisfy this inequality.

Actually the numbers $\frac{2}{3}$ and $\frac{4}{3}$ are not optimal, but $\frac{\mathrm{e}}{4}$ and $\frac{\mathrm{e}}{2}$ are. Being especially careful with the estimates leads to the following result:

Kurlberg, Lagarias, P (2011): *There is a positive constant* c_1 *such that for infinitely many* n *we have*

$$D(n) > 1 - \frac{c_1}{(\log\log n)^{1-\frac{\mathrm{e}}{2}\log 2}(\log\log\log n)^{\frac{1}{2}}}.$$

Note that $1 - \frac{\mathrm{e}}{2}\log 2 = 0.0579153\ldots$.

This is optimal for our method of proof, but is this the optimal result? It turns out that yes, apart from the constant c_1, it is optimal:

Kurlberg, Lagarias, P (2011): *There is a positive constant c_2 such that for all n we have*

$$D(n) < 1 - \frac{c_2}{(\log\log n)^{1-\frac{\mathrm{e}}{2}\log 2}(\log\log\log n)^{\frac{1}{2}}}.$$

The idea for this upper bound: use linear programming!

For a product-free set S in $\mathbb{Z}/n\mathbb{Z}$ and for $d \mid n$, let α_d be the proportion of those $s \in S$ with $\gcd(s, n) = d$ among all residues $r \pmod n$ with $\gcd(r, n) = d$.

Then each α_d is in $[0, 1]$.

Further, if $|S| \geq n/2$, then $\alpha_1 = 0$ and for all u, v with $uv \mid n$, we have

$$\alpha_u + \alpha_v + \alpha_{uv} \leq 2.$$

In some sense, $|S|/n$ is closely modeled by $\sum_{d|n} \alpha_d/d$.

So, the LP is to maximize $\sum_{d|n} \alpha_d/d$ given the above constraints.

Since we already know that $D(n)$ can be fairly large, we need not prove we have found the maximum of the LP, just some upper bound for it. It is known that any feasible solution to the *dual* LP gives an upper bound for the primary LP. Thus, we write down the dual LP, find a fairly trivial feasible solution, and then "shift mass" to make it better.

And, voilà, our upper bound for all n's tightly matches our constructed lower bound for champion n's.

Sated now with products, lets move on to sums ...

No, we're not going to start with addition tables. The analogous problem is trivial, in the addition table for the integers from 1 to N there are precisely $2N - 1$ distinct sums.

But what about *sum-free* sets? Here we have a set of positive integers that contains none of the pairwise sums of its elements. How dense can such a set be?

This too is easy. The odd numbers form a sum-free set of asymptotic density $\frac{1}{2}$. And one cannot do better.

Here's the proof. Say $\mathcal{A}$ is a sum-free set of positive integers and $a \in \mathcal{A}$. Then the set $a + \mathcal{A}$ is disjoint from $\mathcal{A}$. If $\mathcal{A}$ has $N = N(x)$ members in $[1, x]$, then $a + \mathcal{A}$ has $N + O(1)$ numbers here, so $x \geq 2N + O(1)$. Hence for all x, we have $N(x) \leq \frac{1}{2}x + O(1)$. We conclude that the upper density of a sum-free set $\mathcal{A}$ of positive integers is at most $\frac{1}{2}$.

Let us look at a somewhat more subtle problem. How dense can a sum-free subset of $\mathbb{Z}/n\mathbb{Z}$ be?

If n is even, then take the odd residues, and this is best possible.

But what if n is odd?

Diananda & Yap (1969), **Green & Ruzsa** (2005):
If n is solely divisible by primes that are 1 (mod 3), *then the maximal density of a sum-free set in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{1}{3} - \frac{1}{3n}$. If n is divisible by some prime that is* 2 (mod 3), *then the maximal density of a sum-free set in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{1}{3} + \frac{1}{3p}$, where p is the least such prime. Otherwise, the maximal density of a sum-free set in $\mathbb{Z}/n\mathbb{Z}$ is $\frac{1}{3}$.*

This problem has been considered in general finite abelian groups and also for non-abelian groups. A survey article by recent Jeopardy contestant Kiran Kedlaya:

Product-free subsets of groups, then and now, Communicating mathematics, 169–177, Contemp. Math., **479**, Amer. Math. Soc., Providence, RI, 2009.

After hearing a shorter version of this talk at a conference in Georgia last October, several graduate students asked me the following question: What if you consider both sums *and* products?

Well, there is a famous and seminal problem here in which the Erdős multiplication-table theorem plays a role:

Among all sets $\mathcal{A}$ of N positive integers what is the minimum value of $|\mathcal{A}+\mathcal{A}|+|\mathcal{A}\cdot\mathcal{A}|$?

If one takes $\mathcal{A}=\{1,2,\ldots,N\}$, then $|\mathcal{A}+\mathcal{A}|=2N-1$ and $|\mathcal{A}\cdot\mathcal{A}|=N^2/(\log N)^{c+o(1)}$, so for large N,

$$|\mathcal{A}+\mathcal{A}|+|\mathcal{A}\cdot\mathcal{A}|>N^{2-\epsilon}.$$

If on the other hand we take $\mathcal{A} = \{1, 2, \ldots, 2^{N-1}\}$, then $|\mathcal{A} \cdot \mathcal{A}| = 2N - 1$ and $|\mathcal{A} + \mathcal{A}| = \frac{1}{2}N^2 + \frac{1}{2}N$, so that again

$$|\mathcal{A} + \mathcal{A}| + |\mathcal{A} \cdot \mathcal{A}| > N^{2-\epsilon}. \tag{1}$$

Erdős & Szemerédi asked in 1983: Is (1) true for any set $\mathcal{A}$ of N positive integers?

There has been a parade of results getting better and better lower bounds, with game players being the posers Erdős & Szemerédi, then Nathanson, Chen, Elekes, Bourgain, Chang, Konyagin, Green, Tao, Solymosi, ...

Seeing a couple of Fields medalists in this list, with the problem still not solved, is a bit daunting!

But what the grad students asked was about dense sets $\mathcal{A}$ that are simultaneously sum-free and product-free.

For example, take the numbers that are 2 or 3 (mod 5). It is a set of asymptotic density $\frac{2}{5}$ and is both sum-free and product-free. We cannot do better than $\frac{1}{2}$ for the density (considering only the sum-free property), but can we beat $\frac{2}{5}$ for both sum-free and product-free?

Kurlberg, Lagarias, P (2011): *Say $\mathcal{A}$ is sum-free and product-free with upper density $D(\mathcal{A})$.*

1. *If $\mathcal{A} \subset \mathbb{Z}_{>0}$ with least element a, then $D(\mathcal{A}) \leq \frac{1}{2}\left(1 - \frac{1}{5a}\right)$.*

2. *There is a constant $\kappa_1 > 0$, such that if $\mathcal{A} \subset \mathbb{Z}/n\mathbb{Z}$, then*

$$D(\mathcal{A}) \leq \frac{1}{2} - \frac{\kappa_1}{(\log\log n)^{1-\frac{e}{2}\log 2}(\log\log\log n)^{\frac{1}{2}}}.$$

3. *There is a constant κ_2 and infinitely many n such that for some $\mathcal{A} \subset \mathbb{Z}/n\mathbb{Z}$,*

$$D(\mathcal{A}) \geq \frac{1}{2} - \frac{\kappa_2}{(\log\log n)^{1-\frac{e}{2}\log 2}(\log\log\log n)^{\frac{1}{2}}}.$$

Thank You!