MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
EXAM #1

Problem 1. Statement (a) is false: take the sequence $(0, 1, 0, 2, 0, 3, 0, 4, \dots)$, which has the convergent subsequence $(0, 0, 0, 0, \dots)$. Statement (b) is also false: take $a_n = 1/n$ and $b_n = n$; then (a_n) converges (to 0), (b_n) diverges, but $a_n b_n = 1$ so $(a_n b_n)$ converges. Statement (c) is false: we are given only that $|x_n - x_m| < \epsilon$ in the definition of a Cauchy sequence. Statement (d) is true: there is a bijection between the set of functions $f : \mathbb{N} \to \{0, 1\}$ and the set of sequences with entries $(0, 1)$, which is uncountable by Cantor's diagonalization argument. Statement (e) is false: every nonempty bounded subset has a supremum, but not a maximum, e.g. $(0, 1)$.

Problem 2. Let x, y be upper bounds for A, B. Since A, B are bounded above, by the Axiom of Completeness they have least upper bounds, say $s = \sup(A)$ and $t = \sup(B)$. We claim that $\sup(A + B) = s + t$.

Since $a \le s$ for all $a \in A$ and $b \le t$ for all $b \in B$, we have $a + b \le s + t$ for all $a + b \in A + B$, so $s + t$ is an upper bound for $A + B$. Now let $u \in \mathbb{R}$ be an upper bound for $A + B$. Then $a + b \le u$ for all $a \in A$ and $b \in B$. Thus, for all $b \in B$ we have $a \le u - b$ for all $a \in A$; but since $s = \sup(A)$, we must have $s \le u - b$. Thus $s + b \le u$ for all $b \in B$. But then $b \le u - s$ for all $b \in B$, so $t \le u - s$, so $s + t \le u$. Thus $s + t$ is the least upper bound for $A + B$.

Problem 3. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ satisfy $N > 1/\epsilon^2$. Then for $n \ge N$ we have $n \ge N > 1/\epsilon^2$ so

$$\frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = 1/\sqrt{n} < \epsilon$$

hence $\dfrac{\sqrt{n}}{n+1} \to 0$.

Problem 4. Suppose that $x \ne y$. Let $\epsilon = |y - x|/2 > 0$. Then since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $|x_n - y| < \epsilon$. Let $N = \max(N_1, N_2)$. Then for all $n \ge N$, we have

$$|y - x| = |y - x_n + x_n - x| \le |x_n - y| + |x_n - x| < 2\epsilon = |y - x|$$

which is a contradiction. Thus $x = y$.

Problem 5. Since $a_n \to 0$, there exists $N_1 \in \mathbb{N}$ such that $|a_n| = a_n < \epsilon/2$ for all $n \ge N_1$. Let $b_1 = a_{N_1}$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $a_n < \epsilon/4$ for $n \ge N_2$. Let $b_2 = a_{n_2}$ where $n_2 > \max(n_1, N_2)$. In general, let $N_k \in \mathbb{N}$ be such that $a_n < \epsilon/2^k$ for $n \ge N_k$, and let $n_k > \max(n_1, \dots, n_{k-1}, N_k)$. Then the series (b_n) converges by comparison to the geometric series, indeed

$$\sum_{k=1}^{\infty} b_k < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \frac{\epsilon/2}{1 - 1/2} = \epsilon.$$

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