MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
EXAM #2

Name ________________________________

Problem 1. Mark each as true or false. Briefly justify your answer.

(a) If $A \subset \mathbb{R}$ is countable then A is not open.

(b) The intersection of two perfect sets is perfect.

(c) If $f : A \to \mathbb{R}$ is continuous and A is closed then f is uniformly continuous.

(d) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b]) \subset [f(a), f(b)]$.

(e) If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and $f(a) = f(b)$ then there exists $c \in (a, b)$ such that $f'(c) = 0$.

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Problem 2. Let A be a closed set, let $f : A \to \mathbb{R}$ be a continuous function, and let $S = \{x \in A : f(x) \geq 5\}$. Show that S is closed.

Problem 3. Let

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

(a) Show that f is differentiable at $x = 0$ (using the definition).

(b) Is f' continuous on $\mathbb{R}$? Is f' differentiable on $\mathbb{R}$?

Problem 4. Let $f : A \to \mathbb{R}$ be a function. Suppose there exists $\lambda > 0$ such that

$$|f(x) - f(y)| \leq \lambda |x - y|^2$$

for all $x, y \in A$. Show that f is uniformly continuous on A.

Problem 5.

(a) Let $K_1, \ldots, K_n$ be compact. Show that $K_1 \cup K_2 \cup \cdots \cup K_n$ is compact.

(b) Find an infinite collection $K_1, K_2, \ldots$ of compact sets such that $\bigcup_{n=1}^{\infty} K_n$ is not compact.