MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
EXAM #1

Problem 1. Statement (a) is true: A Cauchy sequence is convergent, and a convergent sequence is bounded. (One can also show directly that a Cauchy sequence is bounded.) Statement (b) is true: we have $\max B = \inf A$. Statement (c) is false: the given set is finite, and a countable set is infinite. Statement (d) is true: the harmonic series $\sum 1/n$ diverges but the sequence $\sum 1/n^2$ converges. Statement (e) is true: the rationals are countable but the interval (a, b) is uncountable, so there are uncountably many irrationals left.

Problem 2. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ satisfy $N > 1/\epsilon^2$. Then for $n \geq N$ we have $n \geq N > 1/\epsilon^2$ so $1/\sqrt{n} < \epsilon$ hence

$$\left|\frac{\sqrt{n}}{\sqrt{n}+1} - 1\right| = \left|\frac{\sqrt{n} - (\sqrt{n}+1)}{\sqrt{n}+1}\right| = \frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}} < \epsilon$$

thus $\lim \sqrt{n}/(\sqrt{n}+1) = 1$.

Problem 3. For (a), $\sup A$ exists by the Axiom of Completeness. For any $a \in A$, we have $\sup A$ is an upper bound for A so $\sup A \geq a \geq 0$.

For (b), let $s = \sup A$. We show that $\sqrt{s} = \sup \sqrt{A}$. For any $\sqrt{a} \in \sqrt{A}$ we have $a \in A$ so $a \leq s$, thus $\sqrt{a} \leq \sqrt{s}$, so $\sqrt{s}$ is an upper bound for $\sqrt{A}$. Next, if b is an upper bound for $\sqrt{A}$, so that $\sqrt{a} \leq b$ for all $a \in A$, then $a \leq b^2$ for all $a \in A$, so b^2 is an upper bound for A and hence $s = \sup A \leq b^2$, so $\sqrt{s} \leq b$. Thus $s = \sup \sqrt{A}$.

Problem 4. Let $\epsilon > 0$. Since (b_n) converges, it is bounded; let $M \in \mathbb{R}_{>0}$ be such that $|b_n| \leq M$ for all n. Then by the triangle inequality, we have $|b_n + b| \leq |b_n| + |b| \leq M + |b|$ for all n.

Since $b_n \to b$, there exists $N \in \mathbb{N}$ such $|b_n - b| < \epsilon/(M + |b|)$. Thus, for $n \geq N$ we have

$$|b_n^2 - b^2| = |b_n - b||b_n + b| < \frac{\epsilon}{M + |b|}|b_n + b| \leq \epsilon \frac{M + b}{M + b} = \epsilon$$

so $\lim b_n^2 = b^2$.

Problem 5. Without loss of generality, we may assume (a_n) is increasing. Let (a_{n_k}) be a convergent subsequence: it is bounded, say $|a_{n_k}| \leq M$ for all $k \in \mathbb{N}$. Now for any $n \in \mathbb{N}$, since the sequence $n_1 < n_2 < \dots$ of integers is unbounded, there exists $k \in \mathbb{N}$ such that $n \leq n_k$; but since the sequence is increasing, we then have $a_n \leq a_{n_k} \leq M$. Thus (a_n) is bounded, so (a_n) converges by the Monotone Convergence Theorem.

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