MATH 052: INTRODUCTION TO PROOFS
EXAM #2

Problem 1(a). $\mathcal{P}(A) = \{\emptyset, \{0\}, \{2\}, \{-6\}, \{0,2\}, \{0,-6\}, \{2,-6\}, \{\emptyset, 2, -6\}\}$.

Problem 1(b). Only S_1 is a partition: S_2 has the empty set as a block, S_3 repeats 5 in two blocks, and 6 does not belong to any block in S_4.

Problem 1(c). True!

Problem 1(d). Take $A = \{1,2\}$ and $B = \{1\}$ and let $f(1) = f(2) = 1$.

Problem 1(e). The sum is equal to $(1^2-2)+(2^2-2)+(3^2-2)+(4^2-2) = -1+2+7+14 = 22$.

Problem 2. For the base case, we have $1 = 2^1 - 1$. For the inductive step, suppose that

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

Then

$$1 + 2 + \cdots + 2^{n-1} + 2^n = (2^n - 1) + 2^n = 2(2^n) - 1 = 2^{n+1} - 1.$$

So, by the principal of mathematical inductino, the statement is true for all integers $n \geq 1$.

Problem 3(a). R is reflexive, since $a + a = 2a$ is even for all $a \in \mathbb{Z}$. R is symmetric, because if $a + b = 2k$ is even then so is $b + a = a + b = 2k$. Finally, R is transitive since if $a + b = 2k$ is even and $b + c = 2m$ is even, then $a + c = (a+b) + (b+c) - 2b = 2k + 2m - 2b = 2(k + m - b)$ is even.

Problem 3(b). The equivalence classes are $[1] = \{1,5\} = [5]$, $[2] = \{2,3\} = [3]$ and $[4] = \{4\}$.

Problem 4. We have

$$\begin{aligned}
(x,y) \in A \times (B \cap C) &\Leftrightarrow (x \in A) \text{ and } (y \in B \cap C) \\
&\Leftrightarrow (x \in A) \text{ and } (y \in B \text{ and } y \in C) \\
&\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\
&\Leftrightarrow (x,y) \in A \times B \text{ and } (x,y) \in A \times C \\
&\Leftrightarrow (x,y) \in (A \times B) \cap (A \times C).
\end{aligned}$$

Problem 5. First, we show that f is injective. Let $x_1, x_2 \in \mathbb{R} \setminus \{2\}$ and suppose that $f(x_1) = f(x_2)$. Then

$$\begin{aligned}
f(x_1) &= f(x_2) \\
\frac{3x_1}{x_1 - 2} &= \frac{3x_2}{x_2 - 2} \\
3x_1(x_2 - 2) &= 3x_2(x_1 - 2) \\
3x_1x_2 - 6x_1 &= 3x_1x_2 - 6x_2 \\
6x_1 &= 6x_2 \\
x_1 &= x_2
\end{aligned}$$

So f is injective.

Date: 2 November 2011.

Next, we show that f is surjective. Let $y \in \mathbb{R} \setminus \{3\}$. We solve for x in terms of y:

$$\begin{aligned} f(x) = \frac{3x}{x-2} &= y \\ 3x &= y(x-2) = xy - 2y \\ 3x - xy &= -2y \\ x(y-3) &= 2y \\ x &= \frac{2y}{y-3} \end{aligned}$$

Since $f(2y/(y-3)) = y$, we see that f is surjective.