MATH 052: INTRODUCTION TO PROOFS
REVIEW, FINAL EXAM

Problem 1. Let $A \subseteq S$. Prove that

$$S \setminus (S \setminus A) = A.$$

Solution. First, we prove $\subseteq$. Let $x \in S \setminus (S \setminus A)$. Then, by definition of complement, $x \in S$ and $x \notin (S \setminus A)$. We have $x \notin (S \setminus A)$ if $x \in (S \setminus A)$ is false, which means ($x \in S$ and $x \notin A$) is false; thus ($x \notin S$ or $x \in A$) is true. Since $x \in S$, we must have the latter, and $x \in A$.

Second, we prove $\supseteq$. Let $x \in A$. Then $x \notin S \setminus A$ by definition of complement. Also, $x \in S$ since $A \subseteq S$. Thus, by definition of complement, $x \in S \setminus (S \setminus A)$.

Problem 2. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $c \in \mathbb{R}$ if the following condition holds:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

(a) Write the condition in abbreviated form, using quantifiers.
(b) Write the negation of this condition in a quantified form, using no negation symbols.
(c) Write out part (b) mostly in words.

Solution. For part (a), we have:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon).$$

For part (b), we have

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x)(|x - c| < \delta \wedge |f(x) - f(c)| \geq \epsilon).$$

In negating a nested statement like this, swap $\exists$ and $\forall$; and the negation of an implication $P \Rightarrow Q$ is the conjunction $P \wedge \sim Q$. For part (c), "A function f is not continuous at $c \in \mathbb{R}$ if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in \mathbb{R}$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon$."

Problem 3. Prove by induction that $n! < n^n$ for all integers $n > 1$.

Solution. The base case $n = 2$ is true: indeed, $2! = 2 < 4 = 2^2$.

Next, we prove the inductive step. Suppose that $n! < n^n$; we show $(n + 1)! < (n + 1)^{n+1}$. We have

$$(n + 1)! = (n + 1)n! < (n + 1)n^n < (n + 1)(n + 1)^n = (n + 1)^{n+1}.$$

Thus, by the principle of mathematical induction, we have $n! < n^n$ for all $n > 1$.

Problem 4. Show that $\#\mathbb{Z} \leq \#[0, 1]$.

Solution. By definition, we need to show that there exists an injective map $f : \mathbb{Z} \to [0, 1]$.

The map

$$\begin{aligned} f : \mathrm{N} &\to [0, 1] \\ n &\mapsto 1/n \end{aligned}$$

is an injective map: if $f(n) = 1/n = 1/m = f(m)$ then $m = n$. In class, we showed that $\#\mathbb{Z} = \#\,\mathrm{N}$; indeed, the map

$$\begin{aligned} g : \mathbb{Z} &\to \mathrm{N} \\ n &\mapsto \begin{cases} 1 & \text{if } n = 0; \\ 2n & \text{if } n > 0; \\ 2|n| + 1 & \text{if } n < 0 \end{cases} \end{aligned}$$

is such a bijection.

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Therefore $f \circ g : \mathrm{N} \to [0,1]$ is an injective map, since the composition of two injective maps is injective. Thus $\#\mathbb{Z} \leq \#[0,1]$.

Problem 5. Consider the binary operation $a*b = \dfrac{ab}{3}$ on $\mathbb{Q}\setminus\{0\}$. Show that $*$ is associative and commutative. What is the identity element for $*$?

Solution. $*$ is associative since

$$(a*b)*c = (ab/3)*c = (ab/3)c/3 = abc/9$$
$$a*(b*c) = a*(bc/3) = a(bc/3)/3 = abc/9$$

are equal for all $a, b, c \in \mathbb{Q} \setminus \{0\}$. $*$ is commutative since $a*b = ab/3 = ba/3 = b*a$ for all $a, b \in \mathbb{Q} \setminus \{0\}$.

The identity element $e \in \mathbb{Q} \setminus \{0\}$ is the unique element satisfying $a*e = e*a = a$ for all $a \in \mathbb{Q} \setminus \{0\}$. Well,

$$a*e = e*a = ae/3 = a$$

for all a if and only if $e = 3$, so the identity element is $e = 3$.

Problem 6. Prove that if $a \mid b$ then $a^2 \mid b^2$.

Solution. Suppose that $a \mid b$. Then $b = ca$ for some $c \in \mathbb{Z}$. Then $b^2 = (ca)^2 = (c^2)a^2$, so by definition, $a^2 \mid b^2$.

Problem 7. Let $\sim$ be an equivalence relation on a set S, and let $a, b \in S$. Show that two equivalence classes under $\sim$ are either equal or disjoint, i.e. either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Solution. We proved this in class when working with equivalence relations. Find it in your notes!

See also:

`http://www.emba.uvm.edu/~sands/m52f11/index.html`.