MATH 052: INTRODUCTION TO PROOFS
REVIEW, EXAM #1
SOLUTIONS

Problem 1. Give the truth table for $Q \wedge (P \vee \sim Q)$.

Solution.

P	Q	$P \vee \sim Q$	$Q \wedge (P \vee \sim Q)$
F	F	T	F
F	T	F	F
T	F	T	F
T	T	T	T

Problem 2. What is the negation of the statement "There exists $x \in \mathbb{R}$ such that $x^2 < x$"? Write this in symbols and in words. Is this statement or its negation true?

Solution. The statement is $(\exists x \in R)(x^2 < x)$, so its negation is

$$\sim (\exists x \in R)(x^2 < x) \equiv (\forall x \in \mathbb{R})(x^2 \geq x)$$

which is the statement "For all $x \in \mathbb{R}$ we have $x^2 \geq x$. The statement is true: take $x = 1/2$, for example, then $x^2 = 1/4 < 1/2 = x$.

Problem 3. I got an A+ in every nuclear physics I took. What is the easiest way for this statement to be true?

Solution. It's easy: I've never taken a nuclear physics class! (An implication $P \Rightarrow Q$ is true if the hypothesis P is false.)

Problem 4. List the elements of the sets

$$A = \{n \in \mathrm{N} : n^3 < 100\} \text{ and } B = \{x \in \mathbb{R} : x^2 + 1 = 0\}.$$

Solution. $A = \{1, 2, 3, 4\}$ and $B = \emptyset$.

Problem 5. Let $n \in \mathbb{Z}$. Prove that $(n + 1)^2 - 1$ is even if and only if n is even.

Solution. First we prove the implication $\Leftarrow$, namely, that if n is even then $(n + 1)^2 - 1$ is even. Suppose n is even. Then, by definition, $n = 2k$ for some $k \in \mathbb{Z}$. Therefore

$$(n + 1)^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 2(2k^2 + 2k).$$

Since $2k^2 + 2k \in \mathbb{Z}$, we conclude that $(n + 1)^2 - 1$ is even.

Now we prove the converse implication $\Rightarrow$, namely, that if $(n + 1)^2 - 1$ is even then n is even. Suppose $(n + 1)^2 - 1$ is even. Then $(n + 1)^2 - 1 = 2k$ for some $k \in \mathbb{Z}$. Thus

$$(n + 1)^2 - 1 = n^2 + 2n = 2k$$

so $n^2 = 2(k-n)$. Consequently, n^2 is even. We proved in class that if n^2 is even, then n is even. (Remember? We did this by proving the contrapositive, that if n is odd then n^2 is odd.)

Problem 6. Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.

(a) Determine which of the following are elements of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$. Which are subsets of A?
(b) How many elements are in A?
(c) Determine the sets $\{\emptyset\} \cap A$ and $\emptyset \cup A$.

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Solution. For (a), we have $\emptyset \in A$, $\{\emptyset\} \in A$, but $\{\emptyset, \{\emptyset\}\} \notin A$; also, we have $\emptyset \subseteq A$, $\{\emptyset\} \subseteq A$, and $\{\emptyset, \{\emptyset\}\} \subseteq A$. Tricky!

For (b), there are three elements in A.

For (c), we have $\{\emptyset\} \cap A = \{\emptyset\}$ and $\emptyset \cup A = A$.

Problem 7. Consider the statement

"If a series converges, then its terms go to zero."

This statement is in the form $P \Rightarrow Q$. Write each of the corresponding statements and determine their truth value.

(a) $Q \Rightarrow P$.
(b) $\sim Q \Rightarrow \sim P$.
(c) $\sim P \Rightarrow \sim Q$.
(d) $P \Leftrightarrow Q$.
(e) $\sim (P \Rightarrow Q)$.

Which of these are logically equivalent to the original statement?

Solution. P is the proposition "The series converges" and Q is the proposition "The terms of the series go to zero". (The original statement is true!)

For (a), we have the *converse*: "If the terms of a series go to zero, then it converges." This statement is false, if you remember from calculus.

For (b), we have the *contrapositive*: "If the terms of a series do not go to zero, then it does not converge. This is true, since the original statement is true and the contrapositive is always logically equivalent to the original statement.

For (c), we have: "If a series does not converge, then its terms do not go to zero." This statement is false; probably this is too much calculus for you to remember (good to know, but not on the exam!): an example is the harmonic series $\sum 1/n$.

For (d), we have: "A series converges if and only if its terms go to zero". $P \Leftrightarrow Q$ is the same as $(P \Rightarrow Q)$ and $(P \Leftarrow Q)$; we saw in (a) that the latter is false, so this statement is false.

For (e), remember that an implication $P \Rightarrow Q$ is logically equivalent to $\sim (P \wedge \sim Q)$, so its negation is equivalent to $P \wedge \sim Q$: the only thing that can go wrong is that the hypothesis is true and the conclusion is false. So the answer is: "There exists a series which converges but its terms do not go to zero."

Only the contrapositive is logically equivalent.

Problem 8. Let A, B be sets. Prove that $A \cup B = A$ if and only if $B \subseteq A$.

Solution (Reminder: these problems were rejects from the exam! There is a lot going on in this problem, but nothing you can't handle!). First suppose $A \cup B = A$; we show that $B \subseteq A$. Let $x \in B$. Then $x \in A \cup B$. Since $A \cup B = A$, we have $x \in A$. We have shown $B \subseteq A$.

Now suppose $B \subseteq A$; we show that $A \cup B = A$. To show this equality, we must show that $A \subseteq A \cup B$ and $A \cup B \subseteq A$. The first containment is obvious. For the other containment, suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, we are done; if $x \in B$, then since $B \supseteq A$, we have $x \in A$. In either case, $x \in A$. Thus $A \cup B = A$.

Whew! Did you draw a Venn diagram to convince yourself that this was true?

Problem 9. The statement

"If $x < 1 + \epsilon$ for all $\epsilon > 0$, then $x < 1$"

is false.

(a) Give the negation of the statement.
(b) Prove that the negation of the statement is true.

Solution. This statement is an implication $P \Rightarrow Q$, where P is "$x < 1 + \epsilon$ for all $\epsilon > 0$" (note the difference between the Greek letter epsilon ϵ and the 'is an element of' symbol $\in$) and Q is "$x < 1$".

For part (a), recall the negation of the implication $P \Rightarrow Q$ is the statement $P \wedge \sim Q$. So the negation is "There exists x such that $x < 1 + \epsilon$ and for all $\epsilon > 0$ and $x \geq 1$."

For (b), the negation is true: take the value $x = 1$.

Problem 10. Is the following sentence true or false? "This sentence is false."

Solution. The statement is neither true nor false: it is self-referential, and exhibits the characteristics of Russell's paradox.