MATH 250B: COMMUTATIVE ALGEBRA
HOMEWORK #3

JOHN VOIGHT

Some solutions are omitted or sketched.

Problem 6. *Let M, N be flat. Show that $M \otimes N$ is flat.*

Solution. If $E' \hookrightarrow E$ is injective, then since M is flat, so is the map $E' \otimes M \hookrightarrow E \otimes M$; since N is also flat, so too is the map $(E' \otimes M) \otimes N \hookrightarrow (E \otimes M) \otimes N$. By associativity of the tensor product, we obtain the injection

$$E' \otimes (M \otimes N) \hookrightarrow E \otimes (M \otimes N),$$

which is to say, $M \otimes N$ is flat.

Problem 7. *Let F be a flat R-module, and let $a \in R$ be an element which is not a zerodivisor. Show that if $ax = 0$ for some $x \in F$ then $x = 0$.*

Solution. The map $R \xrightarrow{a} R$ which is multiplication by a is injective since a is not a zerodivisor, by definition. Since M is flat, the map $R \otimes F \xrightarrow{a \otimes 1} R \otimes F$ is also injective. Since $R \otimes F \cong F$, we see that the map $F \xrightarrow{a} F$ is injective, which is what we were to show.

Problem 8. *Prove the following:*

(i) *Let S be a multiplicative subset of R. Then $S^{-1}R$ is flat over R.*
(ii) *A module M is flat over R if and only if the localization $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of R.*
(iii) *Let R be a principal (entire) ring. A module F is flat if and only if F is torsion free.*

Solution. Statement (i) follows from the fact that $S^{-1}E \cong S^{-1}R \otimes_R E$, and if $E' \to E$ is injective, then so is $S^{-1}E' \to S^{-1}E$.

For (ii), note that $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$, so taking $S = R \setminus \mathfrak{p}$, we may apply (i) and Problem 6 to see that $M_{\mathfrak{p}}$ is flat for every prime ideal $\mathfrak{p}$. For the converse, if M is not flat, then there is an injection $E' \to E$ of R-modules such that $E' \otimes M \to E \otimes M$ is no longer an injection; let N be the kernel of this map. There exists a prime $\mathfrak{p}$ such that $N_{\mathfrak{p}} \neq 0$ (for example, choose a maximal ideal containing all elements which annihilate N). Then $M_{\mathfrak{p}}$ is clearly not flat.

For (iii), one direction follows from Problem 7, and for the converse, if F is torsion free then since R is a principal entire ring, F is the direct limit of its finitely generated submodules which are free by III.7.3, hence flat (using Exercise 12 below).

Date: February 13, 2003.
XVI: 6–9, 12, *.

Problem 9. *Let M be an A-module. We say that M is* faithfully flat *if M is flat, and if the functor*

$$T_M : E \mapsto M \otimes_A E$$

is faithful, that is $E \neq 0$ implies $M \otimes_A E \neq 0$. Prove that the following conditions are equivalent:

(i) *M is faithfully flat;*
(ii) *M is flat, and if $u : F \to E$ is a homomorphism of A-modules, $u \neq 0$, then $T_M(u) : M \otimes_A F \to M \otimes_A E$ is also nonzero;*
(iii) *M is flat, and for all maximal ideals $\mathfrak{m}$ of A, we have $\mathfrak{m}M \neq M$; and*
(iv) *A sequence of A-modules $N' \to N \to N''$ is exact if and only if the sequence tensored with M is exact.*

Solution. First, we show (i) $\Rightarrow$ (ii). Let E' be the image of u. Since $u \neq 0$ we know $E' \neq 0$. Then the image of $T_M(u)$ is $M \otimes_A E'$ which is nonzero, since M is faithful.

For (ii) $\Rightarrow$ (iii), take the nonzero homomorphism $u : A \to A/\mathfrak{m}A$; we have $T_M(u) : M \to M/\mathfrak{m}M$, so $M/\mathfrak{m}M \neq 0$, i.e. $\mathfrak{m}M \neq M$.

Now we show (iii) $\Rightarrow$ (i). Suppose that $E \neq 0$; let $x \in E$ be a nonzero element, and let $\mathfrak{m}$ be a maximal ideal containing the annihilator of x; then we have an injection of R-modules

$$x(R/\mathfrak{m}) \hookrightarrow E/\mathfrak{m}E.$$

(If $ax = 0$ then a is in the annihilator of x.) Tensoring with M, we obtain

$$x(R/\mathfrak{m}) \otimes M \hookrightarrow (E/\mathfrak{m}E) \otimes M$$

which becomes

$$x(M/\mathfrak{m}M) \hookrightarrow (E \otimes M) \otimes R/\mathfrak{m}.$$

By (iii), $M/\mathfrak{m}M \neq 0$, so $E \otimes M \neq 0$.

To show (iv) $\Rightarrow$ (i), taking $N' = 0$ we see that M is flat; if $E \otimes M = 0$ then $0 \otimes M \to E \otimes M \to 0 \otimes M$ is exact, so $0 \to E \to 0$ is exact, so $E = 0$.

To conclude, we show that (i) $\Rightarrow$ (iv). If M is flat, then $N' \to N \to N''$ exact implies $N' \otimes M \to N \otimes M \to N'' \otimes M$ exact. Conversely, if

$$N' \otimes M \xrightarrow{f \otimes 1} N \otimes M \xrightarrow{g \otimes 1} N'' \otimes M$$

is exact, then $(\operatorname{img} g / \ker f) \otimes M = 0$, so by (i) we have $\operatorname{img} g / \ker f = 0$, i.e.

$$N' \xrightarrow{f} N \xrightarrow{g} N''$$

is exact.

Problem 12. *Show that the tensor product commutes with direct limits. In other words, if $\{E_i\}$ is a direct family of modules, and M is any module, then there is a natural isomorphism*

$$\varinjlim (E_i \otimes_A M) \cong (\varinjlim E_i) \otimes_A M.$$

Solution (sketch). Show that $(\varinjlim E_i) \otimes_A M$ satisfies the universal property of the direct limit; this is clear on any finite level.

Problem *. *Let $R = k[x, y]$ be the indicated polynomial ring in two variables over a field k. Show that the maximal ideal (x, y) of R is not flat over R.*

Solution. Using Proposition 3.7, we note that it suffices to show that the multiplication map $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}^2$ is not an isomorphism. This follows because

$$x \otimes y - y \otimes x = xy - yx = 0,$$

so the map even fails to be injective.

Note that $x \otimes y - y \otimes x \neq 0$, because there is a R-linear map

$$\begin{aligned} f : \mathfrak{m} \times \mathfrak{m} &\to k[x, y]/(x, y) \\ (a, b) &\mapsto (\partial a/\partial x)(\partial b/\partial y) \end{aligned}$$

with $f(x, y) = 1 \neq 0 = f(y, x)$.