MATH 250B: COMMUTATIVE ALGEBRA
HOMEWORK #4

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Problem 1. *Let E be a finite-dimensional vector space over a field k. Let $x_1, \dots, x_p$ be elements of E such that $x_1 \wedge \dots \wedge x_p \neq 0$, and similarly $y_1 \wedge \dots \wedge y_p \neq 0$. If $c \in k$ and*

$$x_1 \wedge \dots \wedge x_p = c y_1 \wedge \dots \wedge y_p$$

show that $x_1, \dots, x_p$ and $y_1, \dots, y_p$ generate the same subspace.

Solution. This reduces to the fact: $z_1 \wedge \dots \wedge z_r = 0$ if and only if the elements z_i are linearly dependent. After all, a dependence relation, say $z_r = \sum_i a_i z_i$, allows us to write

$$z_1 \wedge \dots \wedge z_r = \textstyle\sum_i z_1 \wedge \dots \wedge (a_i z_i) = 0$$

by the alternating property.

This implies that the elements $x_i \in E$ are linearly independent, as are the $y_j \in E$; suppose that the element y_j is not in the span of the x_i, which is to say, the elements $x_1, \dots, x_p, y_j$ are linearly independent. Then $x_1 \wedge \dots \wedge x_p \wedge y_j \neq 0$, but

$$c y_1 \wedge \dots \wedge y_p \wedge y_j = 0$$

a contradiction. Therefore the subspace generated by the y_j is contained in the span of the x_i; since both are spaces of dimension p, they are equal.

Problem 2. *Let E be a free module of dimension n over the commutative ring R. Let $f : E \to E$ be a linear map. Let $\alpha_r(f) = \operatorname{tr} \bigwedge^r(f)$, where $\bigwedge^r(f)$ is the endomorphism of $\bigwedge^r(E)$ into itself induced by f. We have*

$$\alpha_0(f) = 1, \qquad \alpha_1(f) = \operatorname{tr}(f), \qquad \alpha_n(f) = \det f,$$

and $\alpha_r(f) = 0$ if $r > n$. Show that

$$\det(1 + f) = \sum_{r \geq 0} \alpha_r(f).$$

[Hint: As usual, prove the statement when f is represented by a matrix with variable coefficients over the integers.] Interpret the $\alpha_r(f)$ in terms of the coefficients of the characteristic polynomial of f.

Solution. First assume that $R = k$ is an algebraically closed field. Then we can choose a basis $e_1, \dots, e_n$ for E such that f is given in Jordan canonical form (see

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Theorem 2.4 in §XIV, for example), with diagonal elements $\alpha_1, \dots, \alpha_n$ (not necessarily distinct). Note the choice of basis does not affect the value of $\det(1+f)$. Then

$$\det(1+f) = \prod_{i=1}^n (1+\alpha_i) = \sum_{I \subset \{1,\dots,n\}} \alpha_I$$

where for $I \subset S = \{1, \dots, n\}$ we denote $\alpha_I = \prod_{i \in I} \alpha_i$.

On the other hand, a basis for $\bigwedge^r E$ is given by $e_{i_1} \wedge \dots \wedge e_{i_r}$ for $i_1 < \dots < i_r$; by definition

$$\textstyle\bigwedge^r(f)(e_{i_1} \wedge \dots \wedge e_{i_r}) = f(e_{i_1}) \wedge \dots \wedge f(e_{i_r})$$

so since f in this basis is lower triangular,

$$\operatorname{tr} \textstyle\bigwedge^r(f) = \sum_{i_1 < \dots < i_r} \alpha_{i_1} \dots \alpha_{i_r}.$$

Matching expressions, we obtain the desired result. Since by definition the characteristic polynomial of f is $P_f(t) = \det(tI - f)$, we see that $\det(1+f) = (-1)^n P_f(-1)$. From the Jordan decomposition, we have

$$P_f(t) = \prod_i (t - \alpha_i) = \sum_{r=0}^n a_r t^r = \sum_{r=0}^n (-1)^r t^r \sum_{\substack{I \subset S \\ \#I = n-r}} \alpha_I;$$

matching these two, we find that $\alpha_r(f) = (-1)^r a_{n-r}$.

Now if A is a linear map given by a matrix $A = (x_{ij})_{i,j}$ with indeterminate coefficients over $\mathbb{Z}$, then performing the above computation in an algebraic closure of $\mathbb{Q}(x_{ij})_{i,j}$, we see then that the result holds for A. Since the conclusion is then an equality over $\mathbb{Z}[x_{ij}]$ (concerning the characteristic polynomial), by the homomorphism $\mathbb{Z}[x_{ij}] \to R$ which takes $x_{ij} \mapsto f_{ij}$, where $f = (f_{ij})$, we see that the conclusion holds for any commutative ring R.

Problem 3. *Let E be a finite dimensional free module over the commutative ring R. Let $E^\vee$ be its dual module. For each integer $r \geq 1$, show that $\bigwedge^r E$ and $\bigwedge^r E^\vee$ are dual modules to each other, under the bilinear map such that*

$$(v_1 \wedge \dots \wedge v_r, v_1' \wedge \dots \wedge v_r') \mapsto \det(\langle v_i, v_j' \rangle)$$

where $\langle v_i, v_j' \rangle$ is the value of v_j' on v_i, as usual, for $v_i \in E$ and $v_j' \in E^\vee$.

Solution. Let $f : \bigwedge^r E \times \bigwedge^r E^\vee \to R$ be the above map; we need to show that f is nonsingular, i.e. that the R-linear map

$$\begin{aligned} \phi : \textstyle\bigwedge^r E &\to \operatorname{Hom}_R(\textstyle\bigwedge^r E^\vee, R) \\ v_1 \wedge \dots \wedge v_r &\mapsto \big((v_1' \wedge \dots \wedge v_r') \mapsto \det(\langle v_i, v_j' \rangle)\big) \end{aligned}$$

is an isomorphism. Let $e_1, \dots, e_n$ be a basis for E, and let $e_1', \dots, e_n'$ be the duals, so that

$$\langle e_i, e_j' \rangle = \begin{cases} 1, & i = j; \\ 0, & \text{else.} \end{cases}$$

A basis for $\bigwedge^r E$ is given by wedge products $e_s = e_{s_1,\dots,s_r} = e_{s_1} \wedge \dots \wedge e_{s_r}$. Then the wedges $e_{s_1,\dots,s_r}' = e_{s_1}' \wedge \dots \wedge e_{s_r}'$ form a basis for $\bigwedge^r E^\vee$, and a basis for $\mathrm{Hom}_R(\bigwedge^r E^\vee, R)$ is given by the characteristic functions defined on basis vectors as

$$\chi_s(e_t) = \chi_{s_1,\dots,s_r}(e_t) = \begin{cases} 1, & e_t = e_s; \\ 0, & \text{else.} \end{cases}$$

Inside this notational morass, we now easily compute that in fact $\phi(e_s) = \chi_s$, since $\phi(e_s)(e_s) = 1 = \det(\langle e_{s_i}, e_{s_j}\rangle)_{i,j})$, the determinant of the identity matrix, and otherwise $\phi(e_s)(e_t) = 0$. This shows that ϕ is an isomorphism.

Problem 4. *Notation being as in the preceding exercise, let F be another R-module which is free, finite dimensional. Let $A : E \to F$ be a linear map. Relative to the bilinear map of the preceding exercise, show that the transpose of $\bigwedge^r A$ is $\bigwedge^r({}^tA)$, i.e. is equal to the rth alternating product of the transpose of A.*

Solution. It suffices to verify (see Chapter XIII, §5) that for all $e_s \in \bigwedge^r E$ and $f_s \in \bigwedge^r F^\vee$ that

$$\langle Ae_s, f_t\rangle = \langle e_s, {}^tAf_t\rangle.$$

But after all,

$$\langle \textstyle\bigwedge^r(A)(e_s), f_t\rangle = \det(\langle Ae_{s_i}, f_{t_j}\rangle) = \det(\langle e_{s_i}, {}^tAf_{t_j}\rangle) = \langle e_s, \textstyle\bigwedge^r({}^tA)(f_t)\rangle;$$

the middle equality follows from the fact that ${}^tA : F \to E$ is the transpose of A, and by the preceding exercise, $\langle,\rangle$ is nonsingular.