QUIZ #10: CALCULUS 1A (Stankova)
Wednesday, April 7, 2004
Section 10:00–11:00 (Voight)

Problem 1. *Find two positive numbers whose product is* 4 *and for which the sum of their squares is a minimum.*

SOLUTION. Let x, y be the numbers. Then $xy = 4$, so $y = 4/x$ And the sum of their squares is

$$S(x) = x^2 + y^2 = x^2 + \left(\frac{4}{x}\right)^2 = x^2 + \frac{16}{x^2};$$

we want this to be a maximum, so we find

$$S'(x) = 2x - \frac{32}{x^3} = 0$$

or $2x^4 - 32 = 2(x^4 - 16) = 2(x^2 - 4)(x^2 + 4) = 2(x - 2)(x + 2)(x^2 + 4) = 0$, so $x = 2$. Therefore $y = 2$ as well.

Finally, we verify that this is a local minimum:

$$S''(x) = 2 + \frac{96}{x^4} > 0.$$

QUIZ #10: CALCULUS 1A (Stankova)
Wednesday, April 7, 2004
Section 11:00–12:00 (Voight)

Problem 1. *Consider the cost function*

$$C(x) = 16 + x^3.$$

given in millions of dollars.

(a) *At what production level x is the average cost minimal?*

(b) *Let*

$$p(x) = 36 - 3x - x^2$$

be the demand function. At what production level x is the profit maximal? What is the maximum profit?

Solution. For (a), the average cost is

$$\frac{C(x)}{x} = \frac{16}{x} + x^2$$

so to minimize we compute the derivative

$$\left(\frac{C(x)}{x}\right)' = \frac{-16}{x^2} + 2x = 0$$

so $x^3 = 8$, or $x = 2$. This is indeed a minimum, as we compute that

$$\left(\frac{C(x)}{x}\right)'' = \frac{32}{x^3} + 2 > 0.$$

For (b), the profit is the revenue minus cost, i.e.

$$P(x) = xp(x) - C(x) = 36x - 3x^2 - x^3 - (16 + x^3) = -16 + 36x - 3x^2 - 2x^3$$

so the profit is maximized when

$$P'(x) = 36 - 6x - 6x^2 = -6(x^2 + x - 6) = -6(x-2)(x+3) = 0$$

i.e. $x = 2$. This is indeed a local maximum, since

$$P''(x) = -6 - 12x < 0$$

The maximum profit is

$$P(x) = -16 + 72 - 12 - 16 = 28$$

million dollars.