MATH 110: LINEAR ALGEBRA
HOMEWORK #6 WORKSHEET

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Choose a partner. Between the two of you, choose a problem you will both try to solve. Work on it individually, and then when you have made sufficient progress share your work with your partner. Write together with your partner a nice solution. Then find another pair and see if you can explain your work to them. (When you hear their solution, be picky!)

(1) Label true or false, and *explain*.

(a) For any $m \times n$-matrix A with rank r, there exists an invertible $m \times m$ matrix P and an invertible $n \times n$ matrix Q such that

$$A = P \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} Q$$

where I_r is the $r \times r$ identity matrix.

(b) For any $n \times n$-matrix A with rank r, there exists an invertible $n \times n$ matrix P such that

$$A = P \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} P^{-1}.$$

(c) If A, B are $n \times n$ matrices over a field F, then $\mathrm{rk}(A) = \mathrm{rk}(B)$ implies $\mathrm{rk}(A^2) = \mathrm{rk}(B^2)$.

(d) $\mathrm{rk}(A + B) \le \mathrm{rk}(A) + \mathrm{rk}(B)$.

(e) Let $A \in M_{n\times n}(\mathbb{R})$ be a real $n \times n$-matrix. Suppose there exists a $B \in M_{n\times n}(\mathbb{C})$ with complex coefficients such that $AB = I$. Then B actually has *real* coefficients.

(2) Let A be an $n \times n$ matrix with entries 1 and -1. Show that the integer $\det(A)$ is divisible by 2^{n-1}.

(3) Let $f(x) = (p_1 - x)(p_2 - x)\cdots(p_n - x) \in \mathbb{R}[x]$ and let Δ_n be the determinant of the matrix

$$\begin{pmatrix} p_1 & a & a & \dots & a \\ b & p_2 & a & \dots & a \\ b & b & p_3 & \dots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & p_n \end{pmatrix}$$

where $a, b \in \mathbb{R}$. Show that if $a \neq b$, that

$$\Delta_n = \frac{bf(a) - af(b)}{b - a}.$$

Conjecture a formula when $a = b$.

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