MATH 255: ELEMENTARY NUMBER THEORY
HOMEWORK #5

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4.3: The Chinese Remainder Theorem

Problem 4.3.4(c). Find all the solutions to the following system of linear congruences:

$$x \equiv 0 \pmod 2$$
$$x \equiv 0 \pmod 3$$
$$x \equiv 1 \pmod 5$$
$$x \equiv 6 \pmod 7.$$

Problem 4.3.12. Solve the following ancient Indian problem: If eggs are removed from a basket $2, 3, 4, 5, 6$ at a time, there remain, respectively, $1, 2, 3, 4, 5$ eggs. But if the eggs are removed 7 at a time, no eggs remain. What is the least number of eggs that could have been in the basket?

Problem 4.3.14*. Show that if $a, b, c \in \mathbb{Z}$ have $\gcd(a, b) = 1$, then there is an integer n such that $\gcd(an + b, c) = 1$.

Problem 4.3.A. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integral coefficients. For $m \in \mathbb{Z}_{>1}$, let $\#X(\mathbb{Z}/m\mathbb{Z})$ denote the number of solutions in $\mathbb{Z}/m\mathbb{Z}$ of the congruence

$$f(x) \equiv 0 \pmod m.$$

(a) Prove that if $m = m_1 m_2$, where $\gcd(m_1, m_2) = 1$, then

$$\#X(\mathbb{Z}/m\mathbb{Z}) = \#X(\mathbb{Z}/m_1\mathbb{Z}) \cdot \#X(\mathbb{Z}/m_2\mathbb{Z}).$$

(b) What can you conclude if $\gcd(m_1, m_2) > 1$?

4.4: Solving Polynomial Congruences

Problem 4.4.1. Find all the solutions of each of the following congruences:

(a) $x^2 + 4x + 2 \equiv 0 \pmod 7$
(b) $x^2 + 4x + 2 \equiv 0 \pmod{49}$
(c) $x^2 + 4x + 2 \equiv 0 \pmod{343}$

Problem 4.4.10. How many incongruent solutions are there to the congruence $x^5 + x - 6 \equiv 0 \pmod{144}$?

Problem 4.4.A. Let $k \in \mathbb{Z}_{>0}$.

(a) Show that the product of any k consecutive integers is divisible by $k!$. *[Hint: Use a binomial coefficient.]*

Date: Due Wednesday, 25 February 2009.

(b) Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, let $r \in \mathbb{Z}$. Let $f^{(k)}(x)$ denote the kth derivative of $f(x)$. Show that each coefficient of $f^{(k)}(x)$ is divisible by $k!$. Conclude that for any $r \in \mathbb{Z}$, $f^{(k)}(r)/k!$ is an integer.

Computation 4.4.2*. Find all solutions of $x^9 + 13x^3 - x + 100336 \equiv 0 \pmod{17^9}$.