MATH 351: RIEMANN SURFACES AND DESSINS D'ENFANTS HOMEWORK #4

Let $\sim$ denote the equivalence relation on (subsets of) $\mathbb{C}$ given by $z \sim w$ if and only if $z - w \in \mathbb{Z}$, and let

$$\pi : \mathbb{C} \to \mathbb{C}/{\sim} = \mathbb{C}/\mathbb{Z}.$$

Let $D = \{z \in \mathbb{C} : -1/2 \leq \operatorname{Re} z \leq 1/2\} \subseteq \mathbb{C}$ be given the subspace topology.

Problem 4.1. Show that the natural inclusion

$$D/{\sim} \hookrightarrow \mathbb{C}/\mathbb{Z}$$

is a homeomorphism, where these spaces are given the quotient topology. (This is a way to formulate mathematically the notion of "gluing to get a cylinder".)

Problem 4.2. For each equivalence class in $\mathbb{C}/\mathbb{Z}$, there exists a unique representative $z \in \mathbb{C}$ such that

$$z \in D_0 = \{z \in \mathbb{C} : -1/2 \leq \operatorname{Re} z < 1/2\}.$$

Write each equivalence class with this choice of representative.

Let $0 < \epsilon < 1/4$. For each $[z] \in \mathbb{C}/\mathbb{Z}$ (with $z \in D_0$), let

$$V_z = B(z, \epsilon) \quad \text{and} \quad U_z = \pi(V_z).$$

(a) For all $[z] \in \mathbb{C}/\mathbb{Z}$, show that $\pi|_{V_z} : V_z \to U_z$ is a bijection.

(b) For $[z] \in \mathbb{C}/\mathbb{Z}$, let

$$\phi_z : U_z \to V_z$$

defined by $\phi_z = \pi|_{V_z}^{-1}$. Show that this defines an atlas on $\mathbb{C}/\mathbb{Z}$ by showing that the maps are (holomorphically) compatible, as follows.

Let $[z_1], [z_2] \in \mathbb{C}/\mathbb{Z}$ and suppose that $U_{z_1} \cap U_{z_2} \neq \emptyset$. Write $U_1 = U_{z_1}$, etc. Let $[w] \in U_1 \cap U_2$, and let $w_1 \in [w] \cap V_1$ and $w_2 \in [w] \cap V_2$. We have two cases.

(i) If $|z_1 - z_2| < 1/2$, show that $|w_1 - w_2| < 1$ and conclude $w_1 = w_2$.

(ii) If $|z_1 - z_2| \geq 1/2$ and say $\operatorname{Re} z_1 \leq 0$, show that $|w_1 - w_2| = 1$ and conclude $w_2 = w_1 + 1$.

In either case, conclude that the transition function

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is holomorphic.

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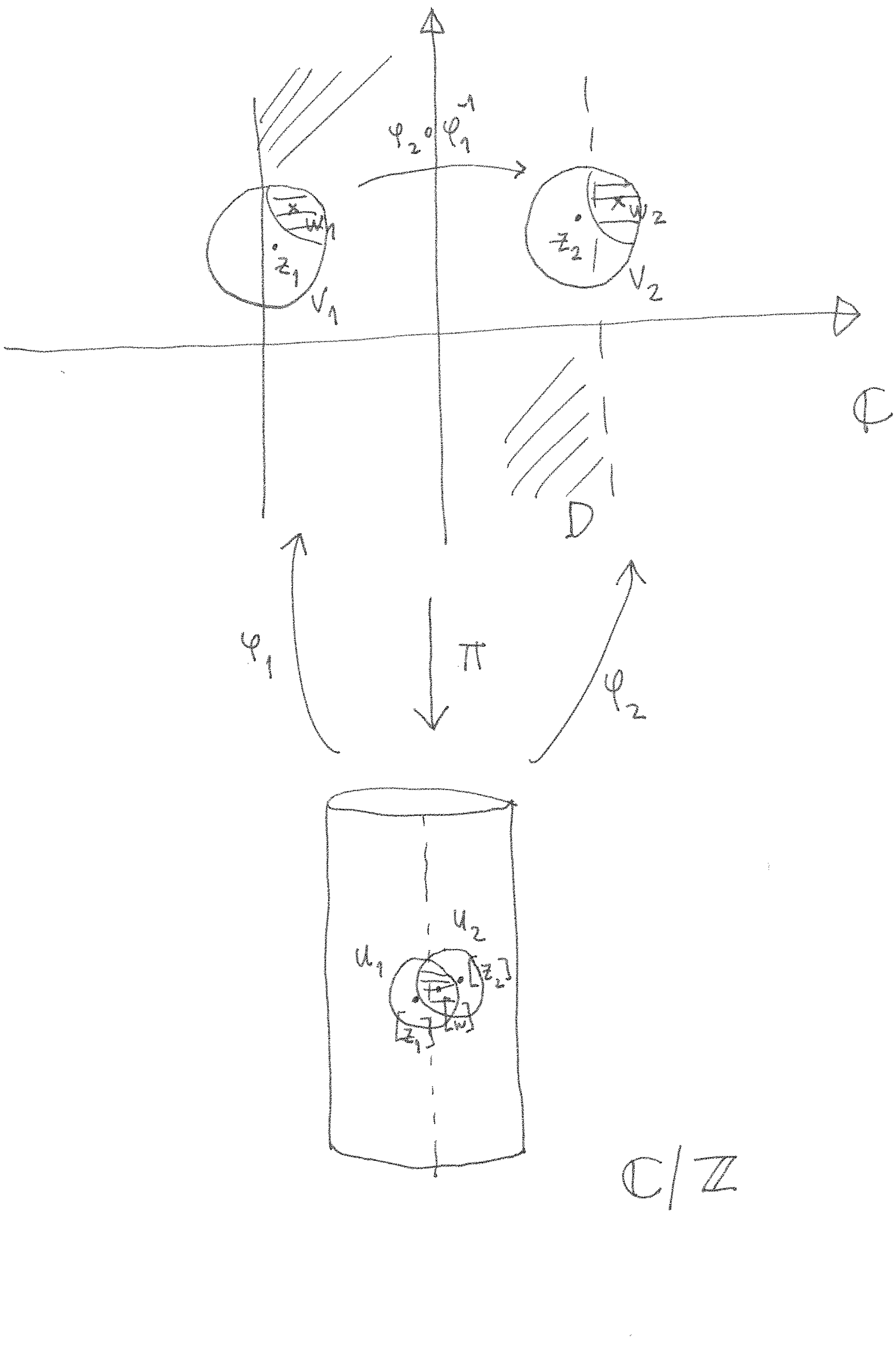

φ₂∘φ₁⁻¹
w₁
z₁
V₁
w₂
z₂
V₂
ℂ
D
φ₁
π
φ₂
u₁
u₂
[z₁]
[z₂]
[w]
ℂ/ℤ