ON THE NONEXISTENCE OF ODD PERFECT NUMBERS

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ABSTRACT. In this article, we show how to prove that an odd perfect number with eight distinct prime factors is divisible by 5.

A perfect number N is equal to twice the sum of its divisors: $\sigma(N) = 2N$. The theory of perfect numbers when N is even is well known: Euclid proved that if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect, and Euler proved that every one is of this type. These numbers have seen a great deal of attention, ranging from very ancient numerology (Saint Augustine considered 6 to be a truly perfect number, since God fashioned the Earth in precisely this many days). They were also very important to the Greeks and to Fermat, whose investigations led him to his little theorem. Today, we have found 38 Mersenne primes (those of the form $2^p - 1$); the latest, found on June 1, 1999 by Nayan Hajratwala, was part of the Great Internet Mersenne Prime Search (GIMPS) (see `http://www.mersenne.org/`); it is

$$2^{6972593} - 1,$$

a number of over two million digits. It is not known if there are infinitely many such primes.

Although the collection of increasingly large perfect numbers continues unabated to this day, no odd perfect number has even been found. Worse still, no proof of their non-existence has been given, either. This article reviews the results concerning odd perfect numbers and shows how to prove that an odd perfect number with eight distinct prime factors must be divisible by 5.

1. Known results

There are a myriad of known conditions that an odd perfect number N must satisfy. Write $N = \prod_{i=1}^{k} p_i^{\alpha_i}$, $p_1 < \cdots < p_k$, $k = \omega(N)$, the number of distinct prime factors of N.

- **Lower bound**: A long computer search by Brent and Cohen (1991) [2] found that $N > 10^{300}$. Heath-Brown (1994) [12] proved that $N < 4^{4^k}$.
- **Multiplicative structure**: Euler proved that $N = \varpi^\gamma m^2$ where $\varpi \equiv \gamma \equiv 1 \pmod 4$. The factor ϖ is often referred to as the special prime. Touchard [24] proved that if N is an odd perfect number, then $N \equiv 1 \pmod 9$ or $N \equiv 9 \pmod{36}$.
- **Large prime factors**: Hagis-Cohen (1998) [10] found that the largest prime factor $p_k \mid N$ satisfies $p_k > 10^6$. Iannucci (1999) [14], [15] extended the work of Pomerance and Hagis proving that $p_{k-1} > 10000$ and $p_{k-2} > 100$. These proofs are almost exclusively computational (so can be easily

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extended with more computing power), and are highly useful in proving other results.

- **Density**: Dickson (1913) [6] proved there are at most finitely many odd perfect numbers with any given number of distinct prime factors. (This also follows from Heath-Brown's result.) Hornfeck and Wirsing (1957) [13] showed that the number of (odd or even) perfect numbers $\leq x$ is $O(x^\epsilon)$ for any $\epsilon > 0$.
- **Number of distinct prime factors**: E.Z. Chein in his doctoral thesis at Penn State (1979) [3] and Hagis (1980) [8] independently proved that the number of distinct prime factors $k = \omega(N) \geq 8$. Kishore (1983) [18] and Hagis (1983) [9] proved that if $3 \nmid N$ then $\omega(N) \geq 11$.
- **Small prime factors**: Grün (1952) [7] and Perisastri (1958) [20] proved that the smallest factor $p_1 \mid N$ satisfies $p_1 < (2/3)k + 2$. Kishore (1981) [17] proved that $p_i < 2^{2^{i-1}}(k - i + 1)$ for $2 \geq i \geq 6$.
- **Exponents**: For $p_i^{\alpha_i} \parallel N$ non-special, let $\alpha_i = 2\beta_i$ for each i. Steuerwald (1937) proved that $\beta_i = 1$ for each i is impossible; Kanold (1942) [16] proved that $\beta_i = 2$ is impossible and that if $d = \gcd(\alpha_i + 1) > 1$, then $9, 15, 21, 33 \nmid d$. There are other results of this type.

While J.J. Sylvester described these numbers as "doubtful or suppositious entities" [23], Descartes was not so pessimistic, giving the spoof example

$$N = 3^2 7^2 11^2 13^2 22021 = 198585576189.$$

Since

$$\begin{aligned}\sigma(N) &= (1+3+3^2)(1+7+7^2)(1+11+11^2)(1+13+13^2)(1+22021) = \\ &= 397171152378 = 2N,\end{aligned}$$

N is perfect if we are willing to ignore the fact that $22021 = 19^2 61$ is not prime. It is also somewhat surprising that for $N = 3^4 7^2 11^2 19^2(-127)$, we have

$$\begin{aligned}&(1+3+3^2+3^3+3^4)(1+7+7^2)(1+11+11^2)(1+19+19^2)(1-127) = \\ &= -44035951806 = 2N,\end{aligned}$$

which is quite close to saying N is perfect if we are willing to ignore the fact that -127 is negative.

2. Basic Results

Let Φ_d be the dth cyclotomic polynomial. If $N = \prod_{i=1}^k p_i^{\alpha_i}$ is perfect, then

$$2N = \sigma(N) = \prod_{i=1}^k \sigma\left(p_i^{\alpha_i}\right) = \prod_{i=1}^k \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = \prod_{i=1}^k \prod_{\substack{d \mid \alpha_i+1 \\ d>1}} \Phi_d(p_i).$$

This most basic result is the backbone of almost every significant result concerning odd perfect numbers, for if $p_i^{\alpha_i} \parallel N$, then every prime factor of $\Phi_d(p_i)$ for $d \mid \alpha_i + 1$ must also divide N.

J.J. Sylvester was perhaps the first to effectively use the following function in the analysis of odd perfect numbers:

Definition 2.1. We define the *abundance* of n to be $h(n) = \sigma(n)/n$.

Proposition 2.2. *Suppose p, q are prime. The function h satisfies:*

(a) *h is multiplicative and is given by*

$$h(n) = \prod_{i=1}^{k} \frac{1 - 1/p^{\alpha+1}}{1 - 1/p};$$

(b) $\lim_{\alpha\to\infty} h(p^\alpha) = h(p^\infty) = p/(p-1)$;
(c) $1 < h(p^\alpha) < h(p^\beta)$ *if* $0 < \alpha < \beta \le \infty$;
(d) $h(p^\alpha) < h(q^\beta)$ *if* $p > q$ *and* $0 < \alpha, \beta \le \infty$; *and*
(e) *N is perfect if and only if* $h(N) = 2$.

The proof is an easy exercise.

Lemma 2.3 (Nagell [19])**.** *If q is a prime that does not divide n, then*

$$\Phi_n(x) \equiv 0 \pmod q$$

is solvable iff $q \equiv 1 \pmod n$. The solutions of the congruence are exactly the numbers that have order n modulo q, and $\Phi_n(x)$ is divisible by exactly the same power of q as $x^n - 1$.

If $q \mid n$, let $n = q^\beta m$, $\gcd(m, q) = 1$. Then

$$\Phi_n(x) \equiv 0 \pmod q$$

is solvable iff $q \equiv 1 \pmod m$. The solutions are the numbers that have order m modulo q, and $\Phi_n(x)$ is divisible by q and not q^2 whenever $n > 2$.

Writing $x^n - 1 = \prod_{d\mid n} \Phi_d(x)$ makes this result clear. Let $v_q(n)$ be the maximal power of q dividing n; we will also write $q^{v_q(n)} \parallel n$. Let $o_q(p)$ be the order of $p \pmod q$. Then we have:

Lemma 2.4. *If $p^\alpha \parallel N$, and $q \ge 3$ is prime, then*

$$v_q(\sigma(p^\alpha)) = \begin{cases} v_q(p^{o_q(p)} - 1) + v_q(\alpha+1), & o_q(p) \mid (\alpha+1) \text{ and } o_q(p) \ne 1; \\ v_q(\alpha+1), & o_q(p) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This result follows from the above and from

$$\sigma(p^\alpha) = \frac{p^{\alpha+1} - 1}{p - 1} = \prod_{\substack{d\mid\alpha+1 \\ d>1}} \Phi_d(p).$$

If $q \mid \sigma(p^\alpha)$, then the relevant values of $d \mid (\alpha+1)$ occur when $d = q^\beta o_q(p)$.

Suppose $o_q(p) \ne 1$ and $o_q(p) \mid \alpha + 1$, the first case of the lemma. Then the only values of d that have $q \mid \Phi_d(p)$ are $o_q(p), q^1 o_q(p), \ldots, q^{v_q(\alpha+1)} o_q(p)$. In the first case, it is divisible by a power of q equal to $q^{o_q(p)} - 1$, and in the others it is divisible only by q. Adding these up gives the result.

If $o_q(p) = 1$, then we are interested in $d = q, q^2, \ldots, q^{v_q(\alpha+1)}$ ($\Phi_1(p)$ is not in the product) and each of these is divisible only by a single q, again the statement.

If $o_q(p) \nmid \alpha + 1$, then p is not a solution to the congruence for any value of d and the product is never divisible by q. □

There is a central role in this theory by Fermat primes, since in the multiplicative group modulo such a prime, no number can have an odd order. Indeed, an argument like the above proves:

Lemma 2.5 ([21])**.** *If q is a Fermat prime,*

$$v_q(\sigma(p^\alpha)) = \begin{cases} v_q(\alpha+1), & p \equiv 1 \pmod q; \\ v_q(p+1) + v_q(\alpha+1), & p \equiv -1 \pmod q,\ p = \varpi; \\ 0, & \textit{otherwise.} \end{cases}$$

We also have:

Proposition 2.6 (Bang, Sylvester [23], Birkhoff-Vandiver [1])**.** *If m is an integer, and $m \geq 2$ then $\Phi_d(m)$ is divisible by a prime q with $o_q(m) = d$, except when $m = 2$ and $d = 1$ or 6 and when m is a Mersenne number and $d = 2$.*

In our case, m will always be an odd prime. Whenever $d = 2$ we have $m \equiv 1 \pmod 4$ so m is not a Mersenne number, so the exceptional cases will be irrelevant.

Corollary 2.7. *If p^α is a component of N and if $d \mid \alpha + 1$, then $\Phi_d(p)$ is divisible by a prime q with $o_q(p) = d$, where $q \equiv 1 \pmod d$.*

Now consider the following general strategy. Suppose we have a Fermat prime q, and $q^\nu \parallel \sigma(p^\alpha)$ for some component p^α of N where $\nu > 0$. Suppose that p is not special. Then from Lemma 2.5, we must have $v_q(\alpha+1) = \nu$ so $\alpha+1 = q^\nu m$, and from the corollary, $\Phi_q(p), \Phi_{q^2}(p), \ldots, \Phi_{q^\nu}(p)$ are each divisible by primes, respectively $r_1, r_2, \ldots, r_\nu$ where the order of p modulo r_i is q^i so that each is distinct, and by Lagrange $r_i \equiv 1 \pmod{q^i}$.

If instead we have the special component ϖ^γ and $q^\nu \parallel \sigma(\varpi^\gamma)$, then from the lemma we must have $\varpi \equiv \pm 1 \pmod q$. If we let $v_q(\gamma+1) = \mu = \nu - v_q(\varpi+1)$, then remembering that $2 \mid \gamma+1$ (γ is the only odd exponent), we write $\gamma+1 = 2q^\mu m$ for some m, so N must be divisible by 2μ distinct primes, where in the corollary we take the sums $\Phi_q(p), \Phi_{2q}(p), \ldots, \Phi_{q^\mu}(p), \Phi_{2q^\mu}(p)$ and corresponding prime divisors $r_1, r_1', \ldots, r_\mu, r_\mu'$ each with distinct orders and like before $r_i \equiv r_i' \equiv 1 \pmod{q^i}$.

It is important to note that the preceding conclusions follow just as easily from the fact that $p \equiv 1 \pmod q$ just by inspection of the q-valuation lemmas. We state this as:

Proposition 2.8. *If $p \equiv 1 \pmod q$ or q is Fermat, then:*

(a) *If p is not special and $q^\nu \parallel \sigma(p^\alpha)$, then N is divisible by distinct primes $r_1, \ldots, r_\nu$ where $r_i \equiv 1 \pmod{q^i}$.*
(b) *If $p = \varpi$ is the special prime, $q^\nu \parallel \sigma(\varpi^\gamma)$ and $q^\mu \parallel (\gamma+1)$, then N is divisible by 2μ distinct primes $r_1, r_1', \ldots, r_\mu, r_\mu'$ where $r_i \equiv r_i' \equiv 1 \pmod{q^i}$.*

3. Two Propositions

Proposition 3.1. *Suppose that q is a Fermat prime and $q^\beta \mid N$. Let ω_q be the maximum number of factors of N which are $1 \pmod q$; suppose k of these factors are unknown, and that ϖ is among the unknown factors. Suppose that at most q^ν divides σ of the known factors and that $\beta - \nu \geq k(\omega_q - 1)$.*

If $\omega_q \leq 2$, then

$$v_q(\varpi+1) \geq \beta - \nu;$$

otherwise,

$$v_q(\varpi+1) \geq \beta - \nu - (k-1)(\omega_q - 2) - \lfloor(\omega_q - 2)/2\rfloor.$$

In either case, ϖ is at least equal to the smallest prime

$$\varpi \equiv -1 \pmod{q^{v_q(\varpi+1)}},$$

so in particular

$$\varpi \geq 2q^{v_q(\varpi+1)} - 1.$$

Proof. By Lemma 2.5, q divides $\sigma(p^\alpha)$ when p is non-special only if $p \equiv 1 \pmod{q}$, and the exponent is exactly $v_q(\alpha+1)$. With q^ν maximally dividing the known factors, we must have $q^{\beta-\nu}$ dividing σ of the unknown factors. At most we have $q^{\omega_q-1} \parallel \sigma(p_i^{\alpha_i})$ for each such i, since by Proposition 2.8, if q^{ω_q} divides such a component then there are at least ω_q other distinct prime factors dividing N congruent to $1 \pmod{q}$, a contradiction. So $q^{\beta-\nu-(k-1)(\omega_q-1)} \mid \sigma(\varpi^\gamma)$, the special prime.

Now if $\varpi \equiv 1 \pmod{q}$, then $v_q(\varpi^\gamma) = v_q(\gamma+1) = \mu$, and we know $2\mu \leq \omega_q - 1$ again counting primes $1 \pmod{q}$. Weakening this slightly, we can say $\mu < \omega_q - 1$, and adding these up, we have $\beta - \nu \geq k(\omega_q - 1) > (k-1)(\omega_q-1) + \mu$, a contradiction since we then have factors of q unaccounted for.

So $\varpi \equiv -1 \pmod{q}$, and we can strengthen the above analysis to get at most $q^{\omega_q-2} \parallel \sigma(p^\alpha)$ (since the input guess of ω_q was one too high). If $\omega_q \leq 2$, this gives us the fact that we already knew: if $q \mid \sigma(N/\varpi^\gamma)$, and $\varpi \equiv -1 \pmod{q}$, then there are at least two primes congruent to $1 \pmod{q}$, and if this cannot hold, then $q^{\beta-\nu} \parallel \sigma(\varpi^\gamma)$.

Otherwise, $\mu = v_q(\gamma+1) < (\omega_q-1)/2$, so since μ is an integer, $\mu \leq \lfloor(\omega_q-2)/2\rfloor$ and at most $q^{\lfloor(\omega_q-2)/2\rfloor} \mid (\gamma+1)$, so

$$v_q(\varpi+1) = \beta - \nu - (k-1)(\omega_q-2) - \lfloor(\omega_q-2)/2\rfloor,$$

with appropriate replacements when $\omega_q < 2$. Remembering that ϖ is odd we must have

$$\varpi + 1 \geq 2q^{v_q(\varpi+1)}$$

as claimed. □

As a direct consequence, we have the special case:

Corollary 3.2. *Suppose each of the conditions in Proposition* 3.1 *is satisfied, and we have $q^\nu \parallel \sigma(N/\varpi^\gamma)$—i.e., $k=1$. Then*

$$\varpi \geq 2q^{\beta-\nu-\lfloor(\omega_q-2)/2\rfloor} - 1.$$

In the case that the Fermat prime $q = 3$, we can also get a very large value for q_7:

Corollary 3.3. *Suppose in the proposition we have either $q = 3$ or $q^\beta \parallel \varpi + 1$. Let P be the largest prime factor dividing $\sigma(q^\beta)$. Then $p_{k-1} \geq \min(\varpi, P)$.*

Proof. Suppose $q = 3$. Then since $\varpi \equiv -1 \pmod{3}$, but $\varpi \equiv 1 \pmod{4}$, we have

$$\left(\frac{3}{\varpi}\right) = \left(\frac{\varpi}{3}\right) = -1,$$

using the Legendre symbol; but then $\varpi \mid \sigma(3^\alpha)$ says $3^{\alpha+1} \equiv 1 \pmod{\varpi}$ and since $\alpha+1$ is odd, we can multiply both sides by 3 and get that 3 is a square, a contradiction. Therefore, $P \neq \varpi$, and the result follows.

The second case follows from

$$\varpi + 1 \geq 2q^\beta > \sigma(q^\beta)$$

so that $P \neq \varpi$. □

This is a useful result because:

Lemma 3.4 (Hagis and McDaniel [11])**.** *Let $\alpha+1$ be an odd prime, and let P be the largest prime factor of $\Phi_{\alpha+1}(q)=\sigma(q^\alpha)$. Then:*

(a) *If $q=3$, then $P\geq 100129$ except for $\alpha\in\{2,4,6,10,16\}$;*
(b) *If $q=5$, then $P\geq 100129$ except for $\alpha\in\{2,4,6\}$; and*
(c) *If $q=17$, then $P\geq 88741$ except for $\alpha=2$.*

I improved this in the case $q=3$:

Lemma 3.5. *If in the conditions of Lemma 3.4 we have $q=3$, then in fact $P\geq 363889$.*

Proof. For each prime $p=\alpha+1$, it suffices to look at primes $q\equiv 1 \pmod p$ such that $q<363889/2$ (by Corollary 2.7) and such that $3^p\equiv 1 \pmod{q^\beta}$ for some $\beta>0$ (by Lemma 2.5). If the product of the q^β is equal to $\sigma(3^{p-1})=\Phi_p(3)$, then this factors completely under 363889; otherwise, it has a larger prime factor. Obtaining this result then reduces to a number of power computations in modular arithmetic. □

In the case that the special prime is one of the known factors, we can argue as follows:

Proposition 3.6. *Suppose that q is a Fermat prime and $q^\beta \mid N$. Let ω_q be the maximum number of prime factors of N which are $1 \pmod q$; suppose that k factors of these are unknown, and that ϖ is among the* known *factors. Suppose that at most q^ν divides σ of the known factors. Then at most $q^{k(\omega_q-1)+\nu}\parallel N$.*

Proof. Just like in Proposition 3.1, we have $q^{\beta-\nu}$ dividing the known factors, and at most we have $q^{\omega_q-1}\parallel\sigma(p_i^{\alpha_i})$ for an unknown factor p_i. Hence at most $q^{k(\omega_q-1)}$ divides the unknown components, and we have the result since $\beta-\nu=k(\omega_q-1)$. □

4. Conclusion

From these propositions, we can extend the work of Hagis and Chein, and prove:

Theorem 4.1. *If N is perfect and $\omega(N)=8$, then $5\mid N$.*

The proof is an extended computation, the results of which are available from the author. As an illustration, we include a portion of the proof.

We know from Kishore [18] and Hagis [9] that $3\mid N$, from Hagis-Cohen [10] that $p_8>10^6$, and from Iannucci [14], [15] that $p_6>100$, $p_7>10000$. First:

Lemma 4.2. *Either $p_2=5$ and $p_3\leq 47$, $p_2=7$ and $p_3\leq 19$, or $p_2=11$ and $p_3=13$.*

Proof. Recall the properties of the abundance function h listed in Proposition 2.2. Suppose $5,7,11\nmid N$. Then

$$h(N)<h(3^\infty 13^\infty 17^\infty 19^\infty 23^\infty 101^\infty 10007^\infty 1000003^\infty)<2,$$

a contradiction. Hence $p_2\leq 11$. In each of these cases,

$$h(3^\infty 5^\infty 53^\infty 59^\infty 61^\infty 101^\infty 10007^\infty 1000003^\infty)<2$$
$$h(3^\infty 7^\infty 23^\infty 29^\infty 31^\infty 101^\infty 10007^\infty 1000003^\infty)<2$$
$$h(3^\infty 11^\infty 17^\infty 19^\infty 23^\infty 101^\infty 10007^\infty 1000003^\infty)<2$$

gives an upper bound on p_3. □

Lemma 4.3. *If* $p_2 = 11$*, then* $p_5 = 19$*.*

Proof. We have $p_3 = 13$, and

$$h(3^\infty 11^\infty 13^\infty 19^\infty 23^\infty 101^\infty 10007^\infty 1000003^\infty) < 2$$

implies $p_4 = 17$ and

$$h(3^\infty 11^\infty 13^\infty 17^\infty 29^\infty 101^\infty 10007^\infty 1000003^\infty) < 2$$

implies $p_5 = 19$ or 23. We have two cases.

First suppose $p_5 = 23$. From

$$h(3^\infty 11^\infty 13^\infty 17^\infty 23^\infty 149^\infty 10007^\infty 1000003^\infty) < 2$$

we have $p_6 \le 139$. Now

$$h(3^4 11^\infty 13^\infty 17^\infty 23^\infty 101^\infty 10007^\infty 1000003^\infty) < 2$$

so $3^6 \mid N$. But $\Phi_7(3) = \sigma(3^6) = 1093$, and

$$h(3^6 11^\infty 13^\infty 17^\infty 23^\infty 1093^\infty 10007^\infty 1000003^\infty) < 2$$

so $3^8 \mid N$. We have $11^4 13^4 \mid N$ since $5 \mid \Phi_3(11)$ and $7 \mid \Phi_2(13) = 13+1$, $61 \mid \Phi_3(13)$ are too small to occur. Also,

$$h(3^\infty 11^\infty 13^\infty 17^1 23^\infty 101^\infty 10007^\infty 1000003^\infty) < 2$$

implies $17^2 \mid N$, and since $\Phi_3(17) = 307$,

$$h(3^\infty 11^\infty 13^\infty 17^2 23^\infty 307^\infty 10007^\infty 1000003^\infty) < 2$$

implies $17^4 \mid N$. Now

$$h(N) > h(3^8 11^4 13^4 17^4 23^2 131^2) > 2$$

implies $p_6 \ge 137$.

Suppose $137 \mid N$. Suppose $\varpi = 137$. We would like to use Proposition 3.6. We have $k = 2$ unknown factors. Also, at most 3 singly divides σ of the known factors because $\sigma(13^2) = \Phi_3(13)$ has an impossible from the above, and $11, 17, 23, 137 \equiv 2 \pmod 3$, $3 \parallel (137+1)$: recall from the Fermat valuation lemma that $3 \mid \sigma(p^\alpha)$ iff $p \equiv 1 \pmod 3$ and $3 \mid \alpha + 1$. Thus $\nu = 1$, and $\omega_3 = 3$ so by the proposition we have at most $3^{2(3-1)+1} = 3^5 \parallel N$, a contradiction. Now since $7 \mid \Phi_3(23)$, and

$$h(3^8 11^4 13^4 17^4 23^4 137^2) > 2.$$

Thus $p_6 = 139$. We can now rule out $\varpi = 17$ since with $\omega_3 = 4$, $\nu = 3$ ($499 \mid \Phi_3(139)$ and $3^2 \parallel \Phi_2(17)$), we conclude $3^9 \parallel N$, and $1093 = \Phi_7(3)$, $757 \mid \Phi_9(3)$ cannot occur so we have a contradiction. Now since $139 \not\equiv 1 \pmod{17}$, we know from Corollary 3.3 that $p_7 \ge 88741$, and

$$h(3^\infty 11^\infty 13^\infty 17^\infty 23^\infty 139^\infty 88741^\infty 1000003^\infty) < 2,$$

a contradiction. $\square$

One can continue in this way to prove the theorem.

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