$\boxed{\text{Math 112}}$ - GGT - Fall 2015

exercise 27: a) i) Show that $\phi(A)(\mathbb{H}) \subset \mathbb{H} \quad \forall A \in PSL_2(\mathbb{Z})$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathrm{Im}\,\phi(A)(z) = \mathrm{Im}\left(\frac{adz + bd\bar{z}}{|cz+d|^2}\right)$

$$= \underbrace{(ad-bc)}_{=1} \frac{\mathrm{Im}(z)}{|cz+d|^2} = \frac{\mathrm{Im}\,z}{|cz+d|^2}$$

$\Rightarrow (z \in \mathbb{H} \Rightarrow \phi(A)(z) \in \mathbb{H})$ $\square$

ii) ϕ is a group isomorphism

$\square$ ϕ Hom: $\phi(A \cdot B) = \phi(A) \circ \phi(B)$ OK

$\phi([\mathrm{Id}_2]) = \mathrm{id}$ OK

$\square$ ϕ surjective: by def.

$\square$ ϕ injective: $\phi(A) = \mathrm{id} \Leftrightarrow$

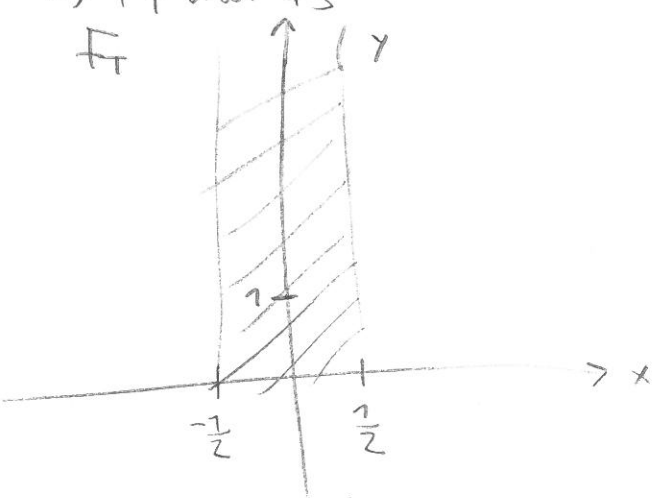
$\forall z \in \mathbb{H} \quad \frac{az+b}{cz+d} = z \Leftrightarrow -cz^2 + (a-d)z + b = 0$

$\Rightarrow (c = b = 0) \wedge ((a = d) \in \{\pm 1\}$, as $\det A = 1)$

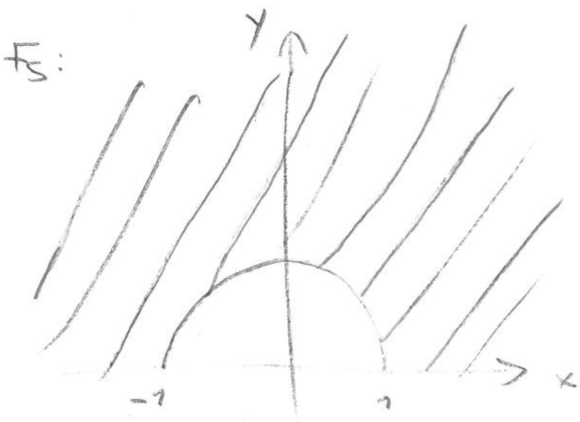
$\Rightarrow A \in [\mathrm{Id}_2]$ $\square$

b) F_T and F_S

F_T



F_S:



proof: F_T: $\phi(T^n)(z) = z + n$, in a similar fashion as for Euclidean tori the fundamental domain is a strip in $\mathbb{H}$ w. identified sides. Topologically F_T is a cylinder. $F_T := \{z \in \mathbb{H} \mid -\frac{1}{2} \leq \mathrm{Re}\,z \leq \frac{1}{2}\}$

F_S: $\phi(S)(z) = -\frac{1}{z}$. A fundamental domain is $F_S := \{z \in \mathbb{H} \mid |z| \geq 1\}$. As ord $S = 2$ we have

- $\phi(S)(F_S) \cup F_S = \mathbb{H}$
- $F_S \cap \phi(S)(F_S) \subset \partial\mathbb{H}$

$\Rightarrow F_S$ fundamental domain

$\mathbb{H}/\langle S \rangle \simeq_{\text{homeo}} \mathbb{R}^2$

c) to show: $\forall z \in \mathbb{H}$ $\exists A \in PSL_2(\mathbb{Z})$, s.th. $\operatorname{Im} \phi(A)(z)$ is maximal

Fix $z \in \mathbb{H}$. Then by a) $\operatorname{Im} \phi(A)(z) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$

As $\binom{c}{d} \in \mathbb{Z}^2 \setminus \{\binom{0}{0}\}$, $cz + d \cdot 1 \in \Lambda \setminus \{0\}$, where

$\Lambda = \mathbb{Z} \cdot z + \mathbb{Z} \cdot 1$ and $|cz+d|^2 \geq \min_{\lambda \in \Lambda \setminus \{0\}} |\lambda|^2 > 0$

$\Rightarrow$ The denominator has a min. and $\operatorname{Im} \phi(A)(z)$ has a max.

d) To show: $F_{T,S} = F_T \cap F_S$

To show: I) $\forall z \in \mathbb{H}$ $\exists A \in \langle T, S \rangle$ s.th. $\phi(A)(z) \in F_{T,S}$

For $z \in \mathbb{H}$, choose $A' \in \langle T, S \rangle$, such that

$\operatorname{Im} \phi(A')(z)$ is maximal and wlog $\phi(A')(z) \in F_T$

(possible as $\operatorname{Im} \phi(T)(z) = \operatorname{Im} z$)

Suppose $\phi(A')(z) \in F_T \setminus F_S$. Then

$|\phi(A')(z)| < 1$ and $\operatorname{Im} \phi(S \cdot A')(z) = \dfrac{\operatorname{Im} \phi(A')(z)}{|\phi(A')(z)|^2} > \operatorname{Im} \phi(A')(z)$ ↯