Math 112 - ggT

Fall 2015 - Assignment 8

exercise 22: Determine a fundamental domain for the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$

to show: $F_{T,S} = F_{PSL_2(\mathbb{Z})}$

We know from exercise 21:

I) $\forall z \in \mathbb{H} \ \exists A \in \langle T,S \rangle \subset PSL_2(\mathbb{Z})$, such that $\phi(A)(z) \in F_{T,S}$

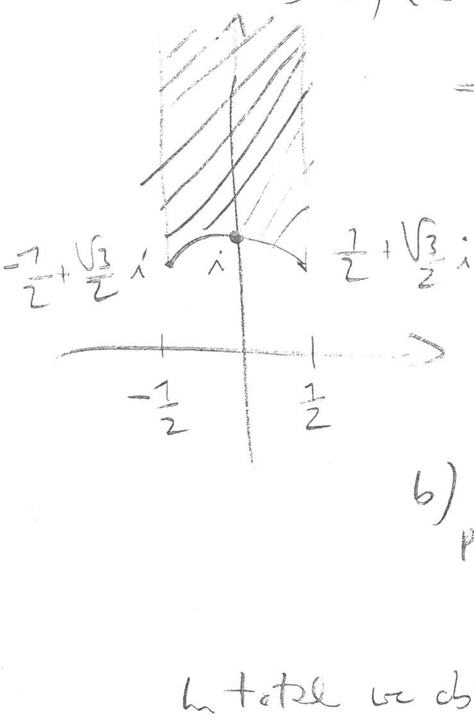
It remains to show: II) $\forall z,w \in F_{T,S}$ where $z \neq w$, we have: $\phi(A)(z) = w$ for an $A \in PSL_2(\mathbb{Z}) \Rightarrow z,w \in \partial F_{T,S}$

proof of II): Let $z,w \in F_{T,S}$, such that $\phi(A)(z) = w$ for some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$

Wlog: $Im(w) \geq Im(z)$. By exercise 21:

1.) $Im(w) = \dfrac{Im(z)}{|cz+d|^2} \geq Im(z) \Rightarrow 1 \geq |c \cdot z + d|^2$

$\Rightarrow$ 2.) $(c \cdot Re(z) + d)^2 + (Im(z) \cdot c)^2 \leq 1$, where $|Re(z)| \leq \frac{1}{2}$, $Im(z) \geq \frac{\sqrt{3}}{2}$



$\Rightarrow \boxed{|c| \leq 1}$

Consider the cases:

a) $c = 0 \xRightarrow{2.)} d = \pm 1 \xRightarrow{1.)} Im(w) = Im(z)$

$\det(A) = 1$

$a = d \Rightarrow A = T^k \xRightarrow{z,w \in F_{T,S}} A = T^{\pm 1} \Rightarrow |Re(z)| = \frac{1}{2}$

$\Rightarrow z,w \in \partial F_{T,S}$

b) $c = \pm 1 \xRightarrow{2.)} \frac{\sqrt{3}}{2} < Im(z) < 1 \xRightarrow{2.)} |d| \leq 1$

pcl functions $\Rightarrow |z| = 1 \xRightarrow{pcl} |cz+d| = 1 \Rightarrow |w| = 1$

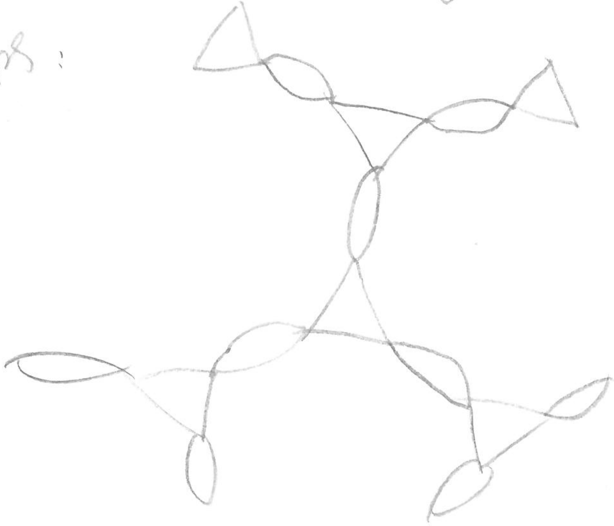
$\Rightarrow z,w \in \partial F_{T,S}$

In total we obtain by I)/II): $F_{\langle T,S \rangle} = F_{PSL_2(\mathbb{Z})}$

and $\langle T,S \rangle = PSL_2(\mathbb{Z})$

exercise 23: a) $\mathbb{Z}_3 * \mathbb{Z}_2 \simeq G := \langle a,b \mid a^3=1, b^2=1\rangle$

Cayley graph:



b) $S\cdot T = P = \begin{pmatrix}0&-1\\1&1\end{pmatrix}$, $S=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. We easily check the conditions

i) for $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $-S\cdot T = P$, hence $T=-P\cdot S^{-1}$

as $S^{-1}=-S$ we have $T\in\langle P,S\rangle$

$\Rightarrow \langle T,S\rangle = \langle P,S\rangle = PSL_2(\mathbb{Z})$

c) $PSL_2(\mathbb{Z})$ operate on $\mathbb{R}\cup\{\infty\}$, where

$\infty = \frac{a}{0}$ $\forall a\in\mathbb{R}\setminus\{0\}$ $\quad$ $0=\frac{a}{\infty}$ $\forall a\in\mathbb{R}\setminus\{0\}$

possible as $\lim_{x\to\pm\infty}\frac{ax+b}{cx+d}=\frac{a}{c}$ for $c\neq 0$, $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in PSL_2(\mathbb{Z})$

As $\phi_A:\mathbb{R}\cup\{\infty\}\to\mathbb{R}\cup\{\infty\}$, $z\mapsto\frac{az+b}{cz+d}$ for $\begin{pmatrix}a&b\\c&d\end{pmatrix}=A\in PSL_2(\mathbb{Z})$

we have that $PSL_2(\mathbb{Z})$ operate also on $\mathbb{R}\cup\{\infty\}=\partial\mathbb{H}$

Set $H_1=\langle P\rangle\simeq\mathbb{Z}_3$, $H_2=\langle S\rangle\simeq\mathbb{Z}_2$

We have that $\phi_P(x)=\frac{-1}{x+1}$ $\phi_{P^2}(x)=-\frac{(x+1)}{x}$ $\phi_S(x)=\frac{-1}{x}$

Set $X_1=\mathbb{R}^-$ and $X_2=\mathbb{R}^+$

Then $\phi_P(X_2)\subset X_1$ $\phi_{P^2}(X_2)\subset X_1$, $\phi_S(X_1)\subset X_2$

Hence by the ping-pong lemma we have:

$PSL_2(\mathbb{Z})\simeq\langle P,S\rangle\simeq\langle P\rangle*\langle S\rangle\simeq\mathbb{Z}_3*\mathbb{Z}_2$

d) $PSL_2(\mathbb{Z})$ operates properly discontinuously on the copies
$(\phi_A \circ F_{T,S})_{A \in PSL_2(\mathbb{Z})}$

Define the Cayley graph:

vertices: $V := (\phi_A \circ F_{T,S})_{A \in PSL_2(\mathbb{Z})}$

edges: $E := \{ (\phi_A \circ F_{T,S}, Q) \mid A \in PSL_2(\mathbb{Z}), Q \in \{T, S\} \}$

As usual: $e = (\phi_A \circ F_{T,S}, Q)$ is an edge between
$\phi_A \circ F_{T,S}$ and $\phi_{A \cdot Q} \circ F_{T,S}$