Math 112: Geometric Group Theory
Fall 2015 - Assignment 4

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keywords: *geodesics in trees, quotients of graphs, products of graphs*

exercise 9. Let $T_3 \subset \mathbb{R}^2$ be a regular triangle. The dihedral group D_3 is the subgroup of isometries of $\mathbb{R}^2$ that maps T_3 onto itself. This group consists of 3 rotations and 3 reflections.

a) Draw a Cayley graph $\Gamma := \Gamma(D_3, \{r, s\})$ of D_3 with a rotation r and a reflection s as a generator.

b) Draw the quotient graphs $\Gamma/\langle r\rangle$ and $\Gamma/\langle s\rangle$ and use **Theorem 2.7** from Chapter III to decide which of the two subgroups $\langle r\rangle$ and $\langle s\rangle$ is not a normal subgroup.

exercise 10. Let $T = (V, E, \delta)$ be a tree, such that $val(v) < \infty$ for all $v \in V$ and let (T^{geom}, d_T) be its geometric realization with the corresponding path metric d_T. Show that

a) for all $x, y \in T^{geom}$ there exists a sequence $\gamma_{x,y} \in S(x, y)$, such that $\ell(\gamma_{x,y}) = d_T(x, y)$.

b) if $\gamma_{x,y}, \gamma'_{x,y} \in S(x, y)$ are two sequences, such that $\ell(\gamma_{x,y}) = \ell(\gamma'_{x,y}) = d_T(x, y)$ then the two images of the segments joining the respective points of the sequences coincide pointwise in T^{geom}.

exercise 11. Let $\Gamma_1 = (V_1, E_1, \delta_1)$ und $\Gamma_2 = (V_2, E_2, \delta_2)$ be two connected, simple graphs without loops and countably many vertices and edges. The *product graph* $\Gamma_1 \times \Gamma_2 = (V, E, \delta)$ is defined in the following way:
The set of vertices V is $V_1 \times V_2$.
The edges E are defined in the following way:
(v_1, v_2) is connected to (w_1, w_2) by an edge $e \in E$, i.e. $(v_1, v_2) \sim (w_1, w_2) \Leftrightarrow$
$(v_1 = w_1$ in Γ_1 and $v_2 \sim w_2$ in $\Gamma_2)$ or $(v_2 = w_2$ in Γ_2 and $v_1 \sim w_1$ in $\Gamma_1)$.

a) Show that $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ can be embedded into $\mathrm{Aut}(\Gamma_1 \times \Gamma_2)$.

b) Is $\mathrm{Aut}(\Gamma_1 \times \Gamma_2) \cong \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$?

c) For $i \in \{1, 2\}$ let G_i be a group and S_i be a generating set, such that $S_i \cap {S_i}^{-1} = \emptyset$. Let $\Gamma_i = \Gamma_i(G_i, S_i)$ be the corresponding Cayley graph. Let $G_1 \times G_2$ be the product group and $(S_1 \times \{1_{G_2}\}) \cup (\{1_{G_1}\} \times S_2)$ be its generating set. Is

$$\Gamma_1 \times \Gamma_2 \cong \Gamma(G_1 \times G_2, (S_1 \times \{1_{G_2}\}) \cup (\{1_{G_1}\} \times S_2))?$$
