Math 2, Winter 2016

Practice Problems for Midterm 1

This is a collection of problems from past exams, and it generally covers the topics that are going to be on your midterm, with one major exception: optimization. Fortunately, Section 3.7 of your textbook has a large number of excellent optimization problems that you should use to practice.

1. Let $f(x) = x^2(x-2)^2$. On what interval(s) is f increasing?

2. Let $f(x) = \frac{1}{x} - \frac{1}{x^2}$. Find the local minimum points, local maximum points, absolute minimum points, and absolute maximum points of f.

3. A 10-foot ladder is leaning against a wall but sliding down, with the bottom of the ladder moving from the wall at $t+1$ feet per second at time t. At time $t = 2$ seconds, the base is 8 feet from the wall. How fast is the top of the ladder falling at this instant?

4. Sketch the graph of $y = x^3 + 3x^2 - 2$. On what intervals is the function increasing? Decreasing? Concave up? Concave down?

5. A ball is thrown into the air, and its height in feet at time t seconds is given by $h(t) = -16t^2 + 64t + 4$. What is the maximum height reached by the ball?

6. Let $f(x) = x^3 - 12x + 17$. Give the intervals of increase/decrease and the intervals of concavity of f.

7. A stone is thrown into a pond, and waves start moving out from it in a circle whose radius is increasing at a rate of 2 m/s. How fast is the area of the circle formed by the waves increasing when the radius of the circle is 5 m?

8. Sketch the graph of $y = \frac{1}{x^2+3}$. On what intervals is the function increasing? Decreasing? Concave up? Concave down?

9. Does the mean-value theorem guarantee that there is some point on the interval $(1, 2)$ where the derivative of $f(x) = \frac{x+1}{x^2}$ is equal to $-\frac{5}{4}$? Choose one:

   A. Yes, because f is continuous on $(1, 2)$.

   B. Yes, because f is differentiable on $(1, 2)$.

   C. Yes, because f is continuous on $[1, 2]$ and differentiable on $(1, 2)$.

   D. No, because f has a discontinuity at $x = 0$.

   E. No, because f does not have a tangent line at $x = \frac{3}{2}$.

10. If a cube's side length is changing at a rate of 5 inches per second, how fast is the volume changing when the cube's side length is 2 inches?

11. Let $f(x) = x^4 - 2x^2$.

    (a) On what interval(s) is f positive? Negative? For what x is $f(x) = 0$?

    (b) On what interval(s) is f increasing? Decreasing?

    (c) On what interval(s) is f concave up? Concave down?

    (d) Sketch the graph of f, labeling the inflection points if there are any.

12. Let $f(x) = x^2 + x + 1$. Find $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2}.$$

13. Find the maximum value of $f(x) = x^3 - 3x + 2$ on the interval $\left[\frac{1}{2}, 4\right]$.

14. Let $f(x) = x^4 + x^2 - 3$. Which of the following describes f in a small interval around $x = 1$ (choose all that apply):

    - f is positive;
    - f is negative;
    - f is increasing;
    - f is decreasing;
    - f is concave up;
    - f is concave down.

15. Let $f(x) = \frac{1}{20}x^5 + \frac{1}{6}x^3 + 1$. Find the inflection point(s) of f.

16. Suppose that the side length of an equilateral triangle is shrinking at a rate of 1 inch per second, so that the figure is always an equilateral triangle. At the moment when the area of the triangle is $400\sqrt{3}$, at what rate is the area of the triangle changing? *Hint: an equilateral triangle with side length s has area $\frac{\sqrt{3}}{4}s^2$.*

17. The area of a square is increasing at a constant rate of 3 m^2/s. How fast is the perimeter of the square increasing when the square has side length 6 m?

18. Let $f(x) = \dfrac{x^2 + 9}{x}$.

    (a) Find the horizontal and vertical asymptote(s) of f.

    (b) Find the interval(s) of increase/decrease of f.

    (c) Find the interval(s) of concavity of f.

    (d) Find the points where f has local minima and local maxima, if it has any.

    (e) Use this information to sketch the graph of f.

19. Find the horizontal and vertical asymptote(s) of $f(x) = \dfrac{x^2 - 4}{x^2 - 2x}$.

20. Find the interval(s) of increase/decrease and the interval(s) of concavity of the function $f(x) = x^3 + 6x^2 + 9x$.

21. Find the maximum value of $(1 + x^2)^{1/2}$ on the interval $[-1, 2]$.

22. Suppose a parked car is leaking oil in such a way that the oil is leaving a very thin circular puddle. If the area of the puddle is growing at a rate of 3 cm^2/hr, how fast is the puddle's radius growing when the puddle has radius 10 cm?