Math 2, Winter 2016

Daily Homework #15 — Solutions

4.2.57. *Use the properties of integrals to verify the inequality without evaluating the integrals:*

$$2 \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le 2\sqrt{2}$$

Solution. On the interval $[-1,1]$, the function $y=\sqrt{1+x^2}$ is lowest at $x=0$ and highest at $x=\pm 1$. Its value at $x=0$ is 1, and its value at $x=\pm 1$ is $\sqrt{2}$. Therefore,

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx \ge \int_{-1}^{1} 1\,dx = 2,$$

and

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx \le \int_{-1}^{1} \sqrt{2}\,dx = 2\sqrt{2}.$$

4.5.2. *Evaluate* $\int x^3\left(2+x^4\right)^5\,dx$ *by substituting* $u=2+x^4$.

Solution. If $u=2+x^4$, then $du=4x^3\,dx$, so

$$\begin{aligned}\int x^3\left(2+x^4\right)^5\,dx &= \tfrac{1}{4}\int \left(2+x^4\right)^5 4x^3\,dx\\ &= \tfrac{1}{4}\int u^5\,du\\ &= \tfrac{1}{4}\cdot\tfrac{1}{6}u^6 + C\\ &= \tfrac{1}{24}\left(2+x^4\right)^6 + C.\end{aligned}$$

4.5.3. *Evaluate* $\int x^2\sqrt{x^3+1}\,dx$ *by substituting* $u=x^3+1$.

Solution. If $u=x^3+1$, then $du=3x^2\,dx$, so

$$\begin{aligned}\int x^2\sqrt{x^3+1}\,dx &= \tfrac{1}{3}\int \sqrt{x^3+1}\cdot 3x^2\,dx\\ &= \tfrac{1}{3}\int \sqrt{u}\,du\\ &= \tfrac{1}{3}\int u^{1/2}\,du\\ &= \tfrac{1}{3}\cdot\tfrac{2}{3}u^{3/2} + C\\ &= \tfrac{2}{9}\left(x^3+1\right)^{3/2} + C.\end{aligned}$$

4.5.7. *Evaluate* $\int x\sin(x^2)\,dx$.

Solution. Set $u = x^2$, so $du = 2x\,dx$. Then

$$\begin{aligned}\int x\sin(x^2)\,dx &= \tfrac{1}{2}\int \sin(x^2)\cdot 2x\,dx \\ &= \tfrac{1}{2}\int \sin(u)\,du \\ &= -\tfrac{1}{2}\cos(u) + C \\ &= -\tfrac{1}{2}\cos(x^2) + C.\end{aligned}$$

4.5.14. *Evaluate* $\int u\sqrt{1-u^2}\,du$.

Solution. Since "u" is already used as a variable, do v substitution: set $v = 1-u^2$, so $dv = -2u\,du$. Then

$$\begin{aligned}\int u\sqrt{1-u^2}\,du &= -\tfrac{1}{2}\int \sqrt{1-u^2}\cdot(-2u)\,du \\ &= -\tfrac{1}{2}\int \sqrt{v}\,dv \\ &= -\tfrac{1}{2}\int v^{1/2}\,dv \\ &= -\tfrac{1}{2}\cdot\tfrac{2}{3}v^{3/2} + C \\ &= -\tfrac{1}{3}\big(1-u^2\big)^{3/2} + C.\end{aligned}$$

4.5.21. *Evaluate* $\int \frac{\cos(x)}{\sin^2(x)}\,dx$.

Solution A. Set $u = \sin(x)$, so $du = \cos(x)\,dx$. Then

$$\begin{aligned}\int \frac{\cos(x)}{\sin^2(x)}\,dx &= \int \frac{1}{\sin(x)^2}\cdot\cos(x)\,dx \\ &= \int \frac{1}{u^2}\,du \\ &= \int u^{-2}\,du \\ &= -u^{-1} + C = -\frac{1}{u} + C = -\frac{1}{\sin(x)} + C.\end{aligned}$$

Solution B. $\int \frac{\cos(x)}{\sin^2(x)}\,dx = \int \frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)}\,dx = \int \csc(x)\cot(x)\,dx$. Recall that the derivative of $\csc(x)$ is $-\csc(x)\cot(x)$, so $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$.