Math 2, Winter 2016

Daily Homework #19 — Solutions

9.3.45. *A tank contains* 1000 *L of brine with* 15 *kg of dissolved salt. Pure water enters the tank at a rate of* 10 *L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after* 20 *minutes?*

Solution. Let $y(t)$ be the amount of salt (in kg) in the tank after t minutes. The water entering the tank has no salt in it. The water leaving the tank has salt concentration $y/1000$ (the amount of salt y divided by the total volume of 1000 L), and it is flowing at a rate of 10 L/min, so the rate at which salt *leaves* the tank is

$$(y/1000) \cdot 10 = 0.01\, y.$$

Therefore, the rate of change of the amount of salt in the tank is $-0.01\, y$. This gives us the following differential equation:

$$\frac{dy}{dt} = -0.01\, y.$$

We immediately recognize this as an equation for exponential decay, whose solution is

$$y = Ae^{-0.01\, t}.$$

We can find A using the initial condition: the tank begins with 15 kg of salt, so $y(0) = 15$. Then

$$15 = Ae^0 \quad \Rightarrow \quad A = 15.$$

Therefore, the amount of salt in the tank after t minutes is

$$y(t) = 15e^{-0.01\, t}.$$

And the amount of salt after 20 minutes is obtained by evaluating this at $t = 20$:

$$y(20) = 15e^{-0.2} \approx 12.3 \text{ kg}.$$

6.5.3. *A bacteria culture initially contains* 100 *cells and grows at a rate proportional to its size. After an hour the population has increased to* 420.

(a) *Find an expression for the number of bacteria after t hours.*

Solution. The culture's growth rate is proportional to its size, so the size is given by the exponential growth function:

$$y(t) = Ae^{kt},$$

for some constants A and k. To find A, we use the initial condition: the culture begins at size 100, so $y(0) = 100$. Then

$$100 = Ae^0 \quad \Rightarrow \quad A = 100.$$

Now the function is

$$y(t) = 100e^{kt}.$$

To find k, we use the fact that the size is 420 after one hour, so $y(1) = 420$. Then

$$420 = 100e^{k\cdot 1} \quad \Rightarrow \quad e^k = 4.2 \quad \Rightarrow \quad k = \ln(4.2).$$

Therefore,

$$y(t) = 100e^{\ln(4.2)\cdot t}.$$

If you want to, you can say that $e^{\ln(4.2)\cdot t} = \left(e^{\ln(4.2)}\right)^t = 4.2^t$, so that

$$y(t) = 100 \cdot 4.2^t.$$

(b) *Find the number of bacteria after* 3 *hours.*

Solution. $y(3) = 100 \cdot 4.2^3 = 7408.8$.

(c) *Find the rate of growth after* 3 *hours.*

Solution. We know that $\frac{dy}{dt} = ky$, as this is the differential equation that gives exponential growth. We found in (a) that $k = \ln(4.2)$, and we found in (b) that $y(3) = 7408.8$, so the rate of growth after 3 hours is

$$\frac{dy}{dt} = ky = \ln(4.2) \cdot 7408.8 \approx 10\,600 \text{ bacteria per hour},$$

or ≈ 2.95 bacteria per second.

(d) *When will the population reach* 10 000*?*

Solution. Set $y(t) = 10\,000$ and solve for t:

$$\begin{aligned} 100e^{\ln(4.2)\cdot t} &= 10\,000; \\ e^{\ln(4.2)\cdot t} &= 100; \\ \ln(4.2) \cdot t &= \ln(100); \\ t &= \frac{\ln(100)}{\ln(4.2)} \approx 3.21, \end{aligned}$$

so the population will reach 10 000 after about 3 hours 13 minutes.

6.5.9. *The half-life of cesium-137 is* 30 *years. Suppose we have a* 100*-mg sample.*

(a) *Find the mass that remains after t years.*

Solution. The mass is given by the exponential decay function:

$$m(t) = m_0e^{kt},$$

for some negative k. The initial mass is $m_0 = 100$, so the function is

$$m(t) = 100e^{kt}.$$

We know that half is left after 30 years, meaning that $m(30) = 50$. We use this to find k:

$$50 = 100e^{k\cdot 30} \quad \Rightarrow \quad e^{k\cdot 30} = 0.5 \quad \Rightarrow \quad 30k = \ln(0.5) \quad \Rightarrow \quad k = \ln(0.5)/30.$$

Therefore,

$$m(t) = 100e^{t\cdot\ln(0.5)/30}.$$

If you want to, you can say that $e^{t\cdot\ln(0.5)/30} = \left(e^{\ln(0.5)}\right)^{t/30} = 0.5^{t/30} = (1/2)^{t/30}$, so that

$$m(t) = 100 \cdot \left(\tfrac{1}{2}\right)^{t/30}.$$

(b) *How much of the sample remains after* 100 *years?*

Solution. $m(100) = 100 \cdot \left(\frac{1}{2}\right)^{100/30} \approx 9.92$ mg.

(c) *After how long will only* 1 *mg remain?*

Solution. Set $m(t) = 1$ and solve for t:

$$\begin{aligned}
100e^{t\cdot\ln(0.5)/30} &= 1; \\
e^{t\cdot\ln(0.5)/30} &= 0.01; \\
t \cdot \ln(0.5)/30 &= \ln(0.01); \\
t &= \frac{30\ln(0.01)}{\ln(0.5)} \approx 199,
\end{aligned}$$

so it will take 199 years for the sample to go down to 1 mg.

6.5.15. *When a cold drink is taken from a refrigerator, its temperature is* 5°C. *After* 25 *minutes in a* 20°C *room its temperature has increased to* 10°C.

(a) *What is the temperature of the drink after* 50 *minutes?*

Solution. Newton's Law of Cooling says that the temperature T follows this differential equation:

$$\frac{dT}{dt} = k(T - 20)$$

(because the temperature of the room is 20). The general solution to this is

$$T(t) = 20 + Ae^{kt}.$$

We find A by using the initial condition $T(0) = 5$:

$$5 = 20 + Ae^0 \quad \Rightarrow \quad A = -15.$$

The function becomes

$$T(t) = 20 - 15e^{kt}.$$

We find k by using the condition $T(25) = 10$:

$$10 = 20 - 15e^{k\cdot 25} \quad \Rightarrow \quad e^{k\cdot 25} = 2/3 \quad \Rightarrow \quad 25k = \ln(2/3) \quad \Rightarrow \quad k = \frac{\ln(2/3)}{25}.$$

Therefore,

$$T(t) = 20 - 15e^{t\cdot \ln(2/3)/25}.$$

After 50 minutes, the temperature is

$$T(50) = 20 - 15e^{50\cdot \ln(2/3)/25} = 13\tfrac{1}{3}$$

(in °C).

(b) *When will its temperature be* 15°C*?*

Solution. Set $T(t) = 15$:

$$\begin{aligned} 20 - 15e^{t\cdot \ln(2/3)/25} &= 15; \\ e^{t\cdot \ln(2/3)/25} &= 1/3; \\ t \cdot \ln(2/3)/25 &= \ln(1/3); \\ t &= \frac{25\ln(1/3)}{\ln(2/3)} \approx 68 \text{ minutes.} \end{aligned}$$