Math 2, Winter 2016

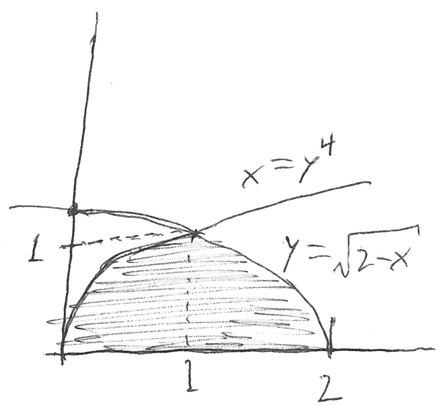
DAILY HOMEWORK #22 — SOLUTIONS

5.1.20. Sketch the region enclosed by

$$x = y^4, \quad y = \sqrt{2-x}, \quad y = 0$$

and find its area.

Solution. ~~We need to find the~~
We express $y = \sqrt{2-x}$
as a function of y, by solving
for x:



$$y^2 = 2 - x$$
$$x = 2 - y^2$$

As $y = \sqrt{2-x} \geq 0$, the domain of $x = 2-y^2$ is $\underline{y \geq 0}$.

Now we need to find the intersection points:

$$x = y^4 \text{ and } x = 2-y^2 \implies y^4 = 2-y^2$$
$$y^4 + y^2 - 2 = 0$$
$$(y^2+2)(y^2-1) = 0$$
$$(y^2+2)(y-1)(y+1) = 0$$

$$\underbrace{y^2+2=0}_{\text{impossible for real } y} \quad \text{or} \quad \underbrace{y-1=0}_{\underline{y=1}} \quad \text{or} \quad \underbrace{y+1=0}_{y=-1 \text{ impossible for } y \geq 0}$$

So they intersect at $y=1$, so y goes from 0 to 1.

The right curve is $x = 2-y^2$, the left curve is $x = y^4$;

so the area is $\int_0^1 ((2-y^2) - y^4)\, dy = \int_0^1 (2 - y^2 - y^4)\, dy$

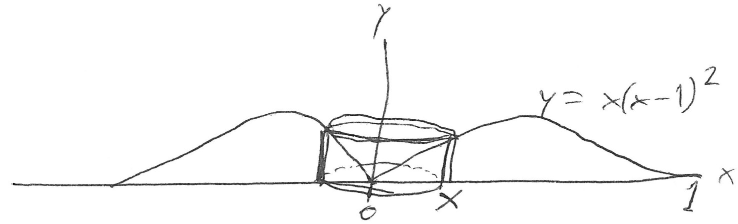
(continued)

$$= \left(2y - \frac{1}{3}y^3 - \frac{1}{5}y^5\right)\Big]_0^1$$
$$= 2 - \frac{1}{3} - \frac{1}{5} = \frac{22}{5}$$

5.3.1. Solution.

It would be hard to use washers because we would have to solve $y = x(x-1)^2$ for x in terms of y, and solving cubic equations is not feasible generally.

The shell is shown at right.



Circumference $= 2\pi x$.

Height $= y = x(x-1)^2$.

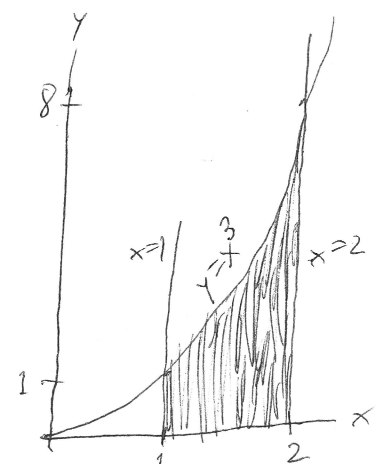
$$\text{Total volume} = \int_0^1 2\pi x \cdot x(x-1)^2\,dx$$
$$= 2\pi \int_0^1 (x^4 - 2x^3 + x^2)\,dx$$
$$= 2\pi \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\right)\Big]_0^1$$
$$= 2\pi\left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = \frac{\pi}{15}.$$

5.3.4. Use cylindrical shells to find the volume of the solid made by revolving the region enclosed by

$$y = x^3,\ y = 0,\ x = 1,\ x = 2$$

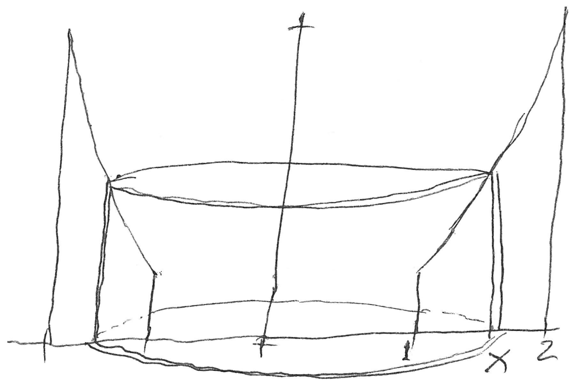
around the y axis.

Solution. The flat region looks like this:



(continued)

Here is a typical shell:



circumference $= 2\pi x$

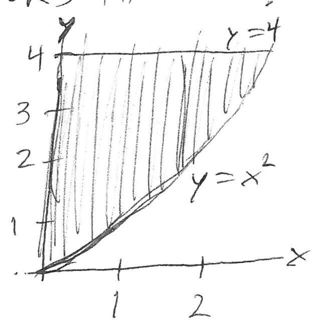
height $= y = x^3$

x goes from 1 to 2, so the total volume is:

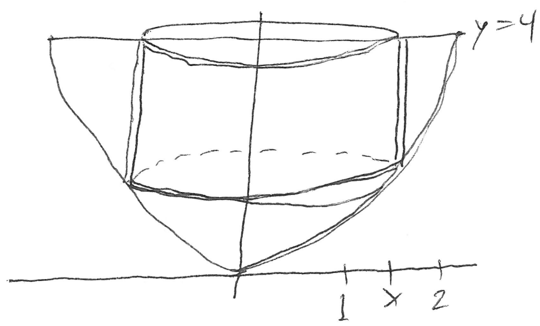
$$\int_1^2 2\pi x \cdot x^3\,dx = 2\pi \int_1^2 x^4\,dx$$
$$= 2\pi \left(\tfrac{1}{5}x^5\right)\Big]_1^2$$
$$= \frac{2\pi}{5}(32-1) = \frac{62\pi}{5}.$$

5.3.5. Same thing but with
$y = x^2$ $(0 \le x \le 2)$, $y = 4$, $x = 0$.

Solution. The flat region looks like this:



Here is a typical shell:



circumference $= 2\pi x$

height $= y_2 - y_1 = 4 - x^2$

x goes from 0 to 2

(because the intersection point is $4 = x^2 \Rightarrow x = 2$).

(continued)

So total volume $= \int_0^2 2\pi x (4-x^2)\, dx$

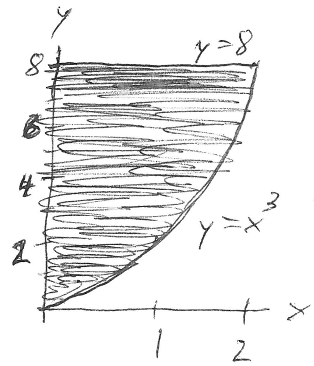
$$= 2\pi \int_0^2 (4x - x^3)\, dx$$

$$= 2\pi \left(2x^2 - \tfrac{1}{4}x^4 \right) \Big]_0^2$$

$$= 2\pi (8-4) = 8\pi.$$

5.3.11. Same thing but revolving around the x axis, with curves $y = x^3$; $y = 8$, $x = 0$.

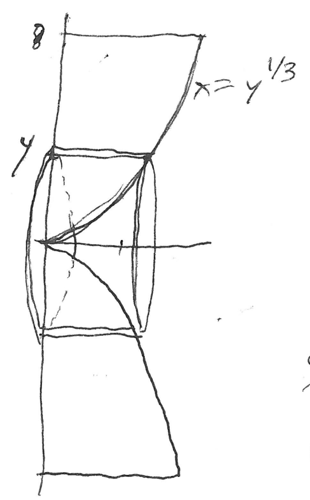
Solution. The flat region looks like this:



We will need to solve for x:

$$y = x^3 \Rightarrow x = y^{1/3}.$$

Here is a typical shell:



radius $= y$
circumference $= 2\pi y$
height $= x = y^{1/3}$.

So the total volume is

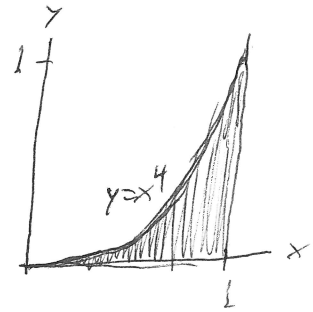
$$\int_0^8 2\pi y \cdot y^{1/3}\, dy$$

$$= 2\pi \int_0^8 y^{4/3}\, dy = 2\pi \cdot \frac{3}{7} y^{7/3} \Big]_0^8$$

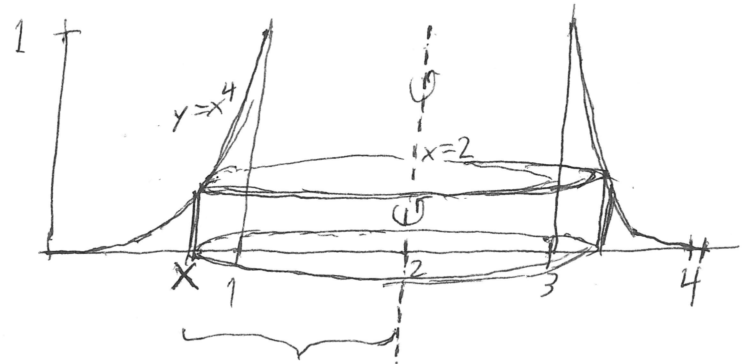
$$= 2\pi \cdot \frac{3}{7} \cdot 128 = \frac{768\pi}{7}.$$

5.3.15. Same thing but revolving around the line $x=2$, with curves $y=x^4$, $y=0$, $x=1$.

Solution. The flat region looks like this:



we are revolving around $x=2$, so this is what a typical cylinder looks like:



radius $= 2-x$

circumference $= 2\pi(2-x)$

height $= y = x^4$

x goes from 0 to 1, so the total volume is

$$\int_0^1 2\pi(2-x)x^4\,dx = 2\pi\int_0^1 (2x^4 - x^5)\,dx$$

$$= 2\pi\left(\frac{2}{5}x^5 - \frac{1}{6}x^6\right)\Big]_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{6}\right) = \frac{7\pi}{15}$$