Math 2, Winter 2016

Daily Homework #24 — Solutions

8.1.7. *Find the exact length of the curve $y = 1 + 6x^{3/2}$, $0 \le x \le 1$.*

Solution. $y' = 9x^{1/2}$, so $(y')^2 = 81x$, so the arc length is

$$\int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+81x} = \tfrac{1}{81}\cdot\tfrac{2}{3}\left(1+81x\right)^{3/2}\Big]_0^1 = \tfrac{2}{243}\left(82^{3/2}-1\right) \approx 6.1$$

8.1.8. *Find the exact length of the curve $y^2 = 4(x+4)^3$, $0 \le x \le 2$, $y > 0$.*

Solution. We can write the curve as $y = 4(x+4)^{3/2}$ (taking the positive square root because $y > 0$). Then $y' = 6(x+4)^{1/2}$, so $(y')^2 = 36(x+4) = 36x + 144$, and the arc length is

$$\int_0^2 \sqrt{1+(y')^2}\,dx = \int_0^2 \sqrt{36x+145} = \tfrac{1}{36}\cdot\tfrac{2}{3}\left(36x+145\right)^{3/2}\Big]_0^2 = \tfrac{1}{54}\left(217^{3/2}-145^{3/2}\right) \approx 26.9$$

8.1.17. *Set up (but do not evaluate) an integral to find the exact length of the curve $y = \ln(1-x^2)$, $0 \le x \le \frac{1}{2}$.*

Solution. $y' = \dfrac{-2x}{1-x^2}$, so the arc length is

$$\int_0^{1/2} \sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,dx.$$

If you want to simplify this, this is what you get:

$$\begin{aligned}
\int_0^{1/2} \sqrt{1+\frac{4x^2}{1-2x^2+x^4}}\,dx &= \int_0^{1/2} \sqrt{\frac{1-2x^2+x^4}{1-2x^2+x^4}+\frac{4x^2}{1-2x^2+x^4}}\,dx \\
&= \int_0^{1/2} \sqrt{\frac{1+2x^2+x^4}{1-2x^2+x^4}}\,dx \\
&= \int_0^{1/2} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\,dx \\
&= \int_0^{1/2} \frac{1+x^2}{1-x^2}\,dx,
\end{aligned}$$

and you can learn how to do this integral in Math 8.

8.1.41. *Find the length of the curve $y = \int_1^x \sqrt{t^3-1}\,dt$, $1 \le x \le 4$.*

Solution. By the Fundamental Theorem of Calculus (part 1),

$$y' = \sqrt{x^3 - 1}.$$

Then the arc length is

$$\int_1^4 \sqrt{1 + (y')^2}\, dx = \int_1^4 \sqrt{1 + (x^3 - 1)}\, dx = \int_1^4 \sqrt{x^3}\, dx = \int_1^4 x^{3/2}\, dx$$
$$= \tfrac{2}{5} x^{5/2}\Big]_1^4 = \tfrac{2}{5}(32 - 1) = \frac{62}{5}.$$

5.2.11 *and* ***5.3.18*** *solutions are on the following two pages.*

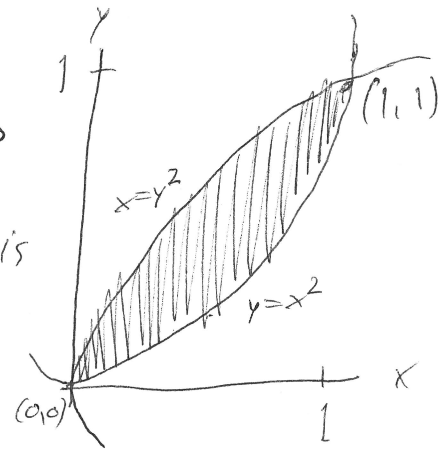
5.2.11. Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = x^2, \quad x = y^2,$$

about the line $y = 1$.

Solution.

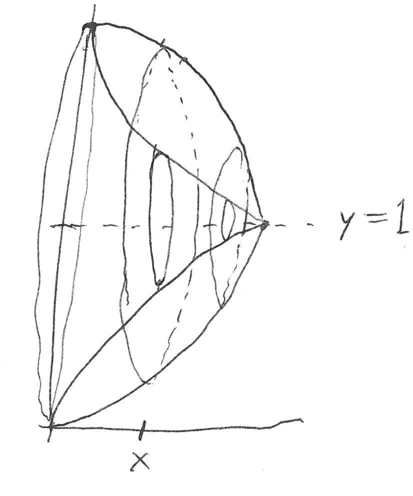
The flat region: ⟶



The curve $x = y^2$ is rewritten as $y = \pm\sqrt{x}$, so the intersection is

$$x^2 = \pm\sqrt{x} \implies x^4 = x$$
$$\implies x(x^3 - 1) = 0$$
$$\implies x = 0, 1.$$

The solid:



Inner radius:

$$r = 1 - \sqrt{x}$$

Outer radius:

$$R = 1 - x^2.$$

So the volume is

$$\int_0^1 (\pi R^2 - \pi r^2)\,dx = \pi \int_0^1 \left((1-x^2)^2 - (1-\sqrt{x})^2\right) dx$$

$$= \pi \int_0^1 (2\sqrt{x} - x - 2x^2 + x^4)\,dx$$

$$= \pi \left(2 \cdot \tfrac{2}{3} x^{\frac{3}{2}} - \tfrac{1}{2} x^2 - \tfrac{2}{3} x^3 + \tfrac{1}{5} x^5\right)\Big]_0^1$$

$$= \pi\left(\left(\tfrac{4}{3} - \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{5}\right) - 0\right) = \frac{11\pi}{30}.$$

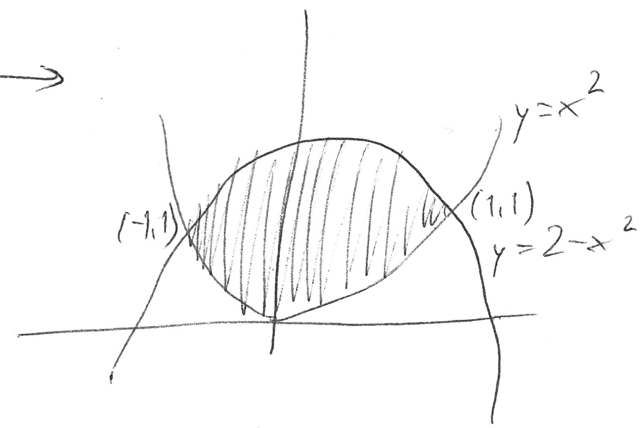
5.3.18. Use cylindrical shells to find the volume generated by rotating the region bounded by the curves

$$y = x^2, \quad y = 2 - x^2$$

about the line $x = 1$.

Solution.

The flat region →



Find the intersection:

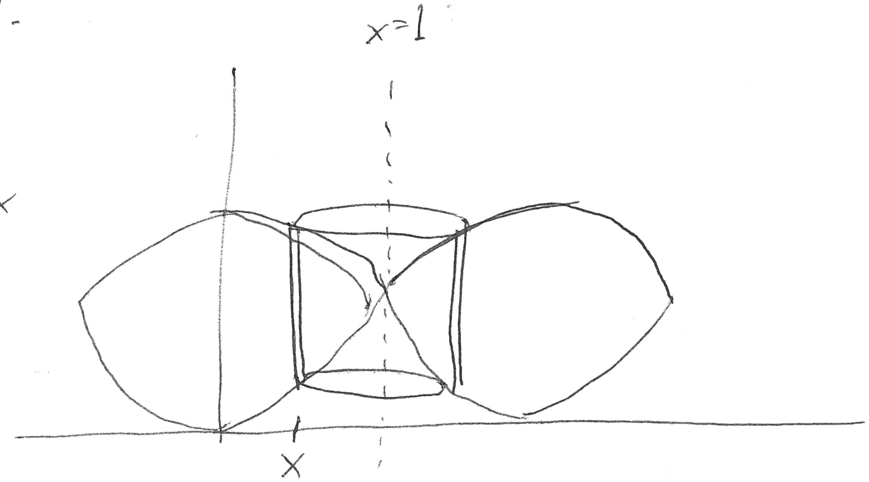
$$x^2 = 2 - x^2$$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1.$$

A typical shell:



radius $= 1 - x$

height $= (2 - x^2) - x^2$

$= 2 - 2x^2$.

So the volume is

$$\int_{-1}^{1} (2\pi r h)\,dx = 2\pi \int_{-1}^{1} (1-x)(2-2x^2)\,dx$$

$$= 2\pi \int_{-1}^{1} (2 - 2x - 2x^2 + 2x^3)\,dx$$

$$= 2\pi \left(2x - x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{2}x^4\right)\Big]_{-1}^{1}$$

$$= \frac{16\pi}{3}.$$