Math 2, Winter 2016

Daily Homework #3 — Solutions

3.1.68. *Show that 5 is a critical number of the function*

$$g(x) = 2 + (x-5)^3$$

but g does not have a local extreme value at 5.

Solution. The derivative is

$$g'(x) = 3(x-5)^2.$$

To find the critical numbers, we set $g'(x) = 0$:

$$3(x-5)^2 = 0; \qquad (x-5)^2 = 0; \qquad x-5 = 0; \qquad x = 5.$$

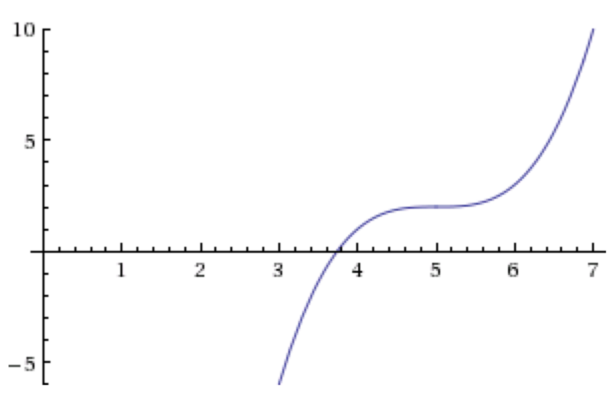
Therefore 5 is a critical number of g. In fact, 5 is the only critical number of g.

To find out whether there is a local extremum (*i.e.* local min or max) at 5, we will use the first-derivative test. (If you try the second-derivative test, it will be inconclusive.) The one critical number is 5, so we can put 4 and 6 into g' to see whether g' is positive or negative before and after 5.

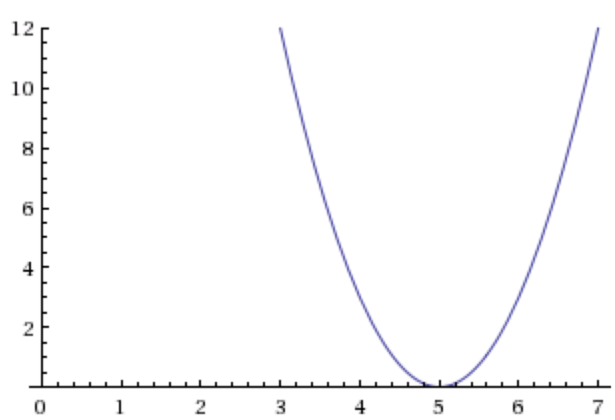
$$g'(4) = 3(4-5)^2 = 3 \cdot (-1)^2 = 3; \qquad g'(6) = 3(6-5)^2 = 3 \cdot 1^2 = 3.$$

This shows that g' is positive before *and* after 5: it does not cross from negative to positive, and it does not cross from positive to negative. So the first-derivative test says g does not have a local extremum at 5.

This is what g and g' look like:



$g(x) = 2 + (x-5)^3$



$g'(x) = 3(x-5)^2$

As you can see, g' does not *cross* the horizontal axis at $x = 5$: it only touches it.

3.2.6. *Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?*

Solution. Since $\tan(0) = 0$ and $\tan(\pi) = 0$, f has the same value at 0 and at π. However, $f'(x) = \sec^2(x)$, which is never zero:

$$\sec^2(x) = 0 \quad \Rightarrow \quad \frac{1}{\cos^2(x)} = 0 \quad \Rightarrow \quad \frac{1}{\cos^2(x)} \cdot \cos^2(x) = 0 \cdot \cos^2(x) \quad \Rightarrow \quad 1 = 0,$$

so $\sec^2(x) = 0$ is impossible.

This does not contradict Rolle's theorem, because f is not continuous on $[0, \pi]$: it has a vertical asymptote at $x = \pi/2$.

3.2.25. *Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?*

Solution. No, such a function cannot exist. This is similar to example 5 in section 3.2. If f is a differentiable function with $f(0) = -1$ and $f(2) = 4$, then by the mean-value theorem there is some number c in the interval $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - (-1)}{2 - 0} = 2.5,$$

so $f'(c) = 2.5 > 2$.

3.2.32. *At 2:00 pm a car's speedometer reads 30 mi/h. At 2:10 pm it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h.*

Solution. Let $v(t)$ be the speed when the time is t past 2:00. Then $v(0 \text{ h}) = 30 \text{ mi/h}$ and $v\left(\frac{1}{6} \text{ h}\right) = 50 \text{ mi/h}$ (since 2:10 is $t = \frac{1}{6}$ h past 2:00), so the average rate of change of v on the interval $\left[0 \text{ h}, \frac{1}{6} \text{ h}\right]$ is

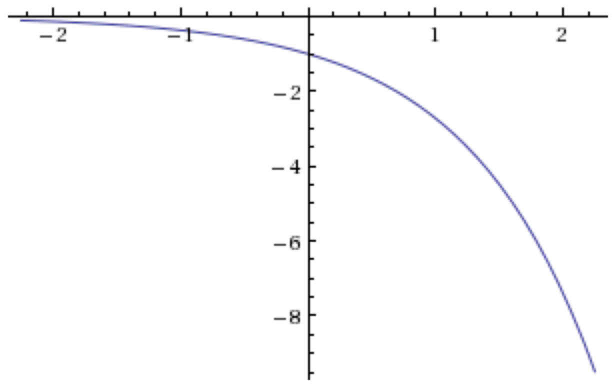
$$\frac{v\left(\frac{1}{6} \text{ h}\right) - v(0 \text{ h})}{\frac{1}{6} \text{ h} - 0 \text{ h}} = \frac{50 \text{ mi/h} - 30 \text{ mi/h}}{\frac{1}{6} \text{ h} - 0 \text{ h}} = 120 \text{ mi/h}^2.$$

So, by the mean-value theorem, there is a time c in $\left(0 \text{ h}, \frac{1}{6} \text{ h}\right)$ at which $v'(c) = 120 \text{ mi/h}^2$.

Now, v is speed as a function of time, so v' is acceleration as a function of time. Therefore, c is a time between 2:00 and 2:10 at which the acceleration is 120 mi/h^2.

3.3.25. *Sketch the graph of a function that satisfies $f'(x) < 0$ and $f''(x) < 0$ for all x.*

Solution. Because $f'(x) < 0$, the function must always be decreasing. Because $f''(x) < 0$, the function must always be concave-downward. One way of thinking is that the derivative must be negative and decreasing, meaning it is getting more and more negative: the function is decreasing steeper and steeper. One such function's graph is shown as a solution in the back of the textbook. Another example is $f(x) = -e^x$, which looks like this:



$f(x) = -e^x$