Math 2, Winter 2016

Daily Homework #5 — Solutions

3.1.51. *Find the absolute maximum and absolute minimum values of* $f(x) = x + \dfrac{1}{x}$ *on the interval* $[0.2, 4]$.

Solution. We can write $f(x) = x + x^{-1}$, so the derivative is

$$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}.$$

To find the critical numbers, we first set $f'(x) = 0$:

$$1 - \frac{1}{x^2} = 0; \qquad 1 = \frac{1}{x^2}; \qquad x^2 = 1; \qquad x = 1 \text{ or } -1.$$

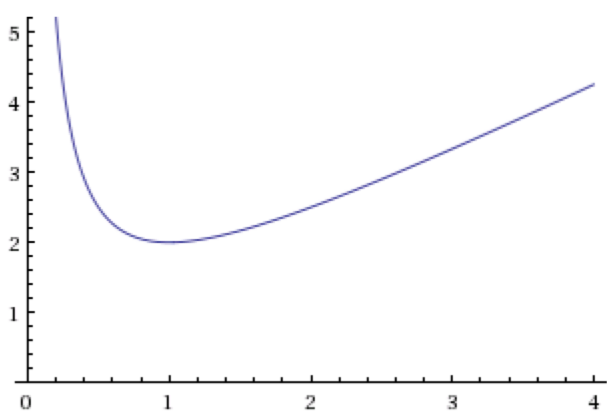
So 1 and -1 are the numbers where f' is zero. We also find where f' is undefined: this happens for $x^2 = 0$, so $x = 0$. The numbers we have found are 1, 0, and -1; but 0 and -1 are not in the domain $[0.2, 4]$, so 1 *is the only critical number of* f *in* $[0.2, 4]$.

To find the (absolute) maximum and minimum, we test the critical points and the endpoints of the domain:

$$f(0.2) = 5.2; \qquad f(1) = 2; \qquad f(4) = 4.25.$$

The highest of these values is 5.2, so the absolute maximum value is $f(0.2) = 5.2$. The lowest of these values is 2, so the absolute minimum value is $f(1) = 2$.

This is what f looks like:



$$f(x) = x + \frac{1}{x}$$

3.1.53. *Find the absolute maximum and absolute minimum values of* $f(t) = t\sqrt{4-t^2}$ *on the interval* $[-1, 2]$.

Solution. We can write $f(t) = t\,(4-t^2)^{1/2}$, and we can take the derivative using the product rule and the chain rule:

$$\begin{aligned}
f'(t) &= (t)' \cdot (4-t^2)^{1/2} + t \cdot \left((4-t^2)^{1/2}\right)' \\
&= 1 \cdot (4-t^2)^{1/2} + t \cdot \tfrac{1}{2}(4-t^2)^{-1/2} \cdot \left(4-t^2\right)' \\
&= 1 \cdot (4-t^2)^{1/2} + t \cdot \tfrac{1}{2}(4-t^2)^{-1/2} \cdot (-2t) = \sqrt{4-t^2} - \frac{t^2}{\sqrt{4-t^2}}.
\end{aligned}$$

To find the critical numbers, we first set $f'(t) = 0$:

$$\begin{aligned}
\sqrt{4-t^2} - \frac{t^2}{\sqrt{4-t^2}} &= 0; \\
\sqrt{4-t^2} &= \frac{t^2}{\sqrt{4-t^2}}; \\
\times\sqrt{4-t^2} \quad & \times\sqrt{4-t^2} \\
4-t^2 &= t^2 \\
2t^2 &= 4 \quad \Rightarrow \quad t^2 = 2 \quad \Rightarrow \quad t = \pm\sqrt{2}.
\end{aligned}$$

So $\sqrt{2}$ and $-\sqrt{2}$ are the numbers where f' is zero. But $-\sqrt{2} \approx -1.4$ is not in the domain $[-1, 2]$, so the only critical number we care about is $\sqrt{2} \approx 1.4$, which *is* in $[-1, 2]$.

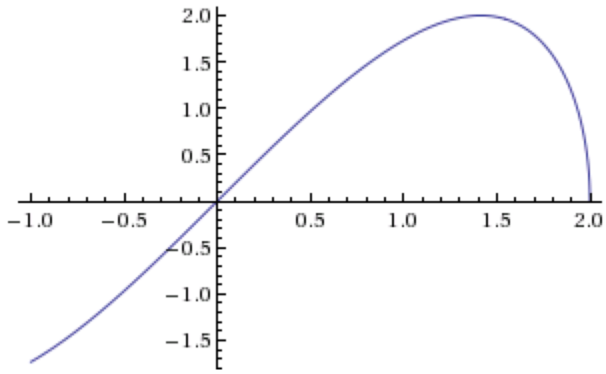
We also find where f' is undefined. The $\sqrt{4-t^2}$ in the denominator is undefined when $4 - t^2 \leq 0$, which happens whenever $t \leq -2$ or $t \geq 2$. The only such t in the domain $[-1, 2]$ is $t = 2$, which we were already going to test because it is an endpoint.

To find the (absolute) maximum and minimum, we test the critical number $\sqrt{2}$ and the endpoints of the domain:

$$f(-1) = -\sqrt{3} \approx -1.7; \qquad f\left(\sqrt{2}\right) = 2; \qquad f(2) = 0.$$

The highest of these values is 2, so the absolute maximum value is $f\left(\sqrt{2}\right) = 2$. The lowest of these values is $-\sqrt{3}$, so the absolute minimum value is $f(-1) = -\sqrt{3}$.

This is what f looks like:



$$f(t) = t\sqrt{4-t^2}$$

3.4.10. *Find* $\lim_{x\to\infty} \dfrac{1-x^2}{x^3-x+1}$ *or show that the limit does not exist.*

Solution A.

$$\lim_{x\to\infty} \frac{1-x^2}{x^3-x+1} = \lim_{x\to\infty} \frac{1/x^2}{1/x^2} \cdot \frac{1-x^2}{x^3-x+1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-\frac{x^2}{x^2}}{\frac{x^3}{x^2}-\frac{x}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-1}{x-\frac{1}{x}+\frac{1}{x^2}}.$$

As $x \to \infty$, the top of this approaches 1 and the bottom approaches ∞, so the whole fraction approaches 0. Therefore, the limit is 0.

Solution B. $\lim_{x\to\infty}(1-x^2) = -\infty$ and $\lim_{x\to\infty}(x^3-x+1) = \infty$, so L'Hôpital's Rule applies:

$$\lim_{x\to\infty} \frac{1-x^2}{x^3-x+1} = \lim_{x\to\infty} \frac{(1-x^2)'}{(x^3-x+1)'} = \lim_{x\to\infty} \frac{-2x}{3x^2-1}.$$

Now $\lim_{x\to\infty}(-2x) = -\infty$ and $\lim_{x\to\infty}(3x^2-1) = \infty$, so again L'Hôpital's Rule applies:

$$\lim_{x\to\infty} \frac{-2x}{3x^2-1} = \lim_{x\to\infty} \frac{(-2x)'}{(3x^2-1)'} = \lim_{x\to\infty} \frac{-2}{6x} = 0.$$

Solution C. In a polynomial, the term that dominates as $x \to \infty$ is the term with the highest power of x. In the top polynomial, this term is $-x^2$; in the bottom polynomial, this term is x^3. So,

$$\lim_{x\to\infty} \frac{1-x^2}{x^3-x+1} = \lim_{x\to\infty} \frac{-x^2}{x^3} = \lim_{x\to\infty} \frac{-1}{x} = 0.$$

3.4.12. *Find* $\lim_{x\to-\infty} \dfrac{4x^3+6x^2-2}{2x^3-4x+5}$ *or show that the limit does not exist.*

Solution A. $\lim_{x\to-\infty} \dfrac{4x^3+6x^2-2}{2x^3-4x+5} =$

$$= \lim_{x\to-\infty} \frac{1/x^3}{1/x^3} \cdot \frac{4x^3+6x^2-2}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{\frac{4x^3}{x^3}+\frac{6x^2}{x^3}-\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{4x}{x^3}+\frac{5}{x^3}} = \lim_{x\to-\infty} \frac{4+\frac{6}{x}-\frac{2}{x^3}}{2-\frac{4}{x^2}+\frac{5}{x^3}}.$$

As $x \to -\infty$, the top of this approaches 4 and the bottom approaches 2, so the whole fraction approaches $4/2 = 2$. Therefore, the limit is 2.

Solution B. $\lim_{x\to-\infty}(4x^3+6x^2-2) = -\infty$ and $\lim_{x\to-\infty}(2x^3-4x+5) = -\infty$, so L'Hôpital's Rule applies:

$$\lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{(4x^3+6x^2-2)'}{(2x^3-4x+5)'} = \lim_{x\to-\infty} \frac{12x^2+12x}{6x^2-4}.$$

Now $\lim_{x\to-\infty}(12x^2+12x) = \infty$ and $\lim_{x\to-\infty}(6x^2-4) = \infty$, so again L'Hôpital's Rule applies:

$$\lim_{x\to-\infty} \frac{12x^2+12x}{6x^2-4} = \lim_{x\to-\infty} \frac{(12x^2+12x)'}{(6x^2-4)'} = \lim_{x\to-\infty} \frac{24x+12}{12x}.$$

Now $\lim_{x\to-\infty}(24x+12) = -\infty$ and $\lim_{x\to-\infty} 12x = -\infty$, so again L'Hôpital's Rule applies:

$$\lim_{x\to-\infty} \frac{24x+12}{12x} = \lim_{x\to-\infty} \frac{(24x+12)'}{(12x)'} = \lim_{x\to-\infty} \frac{24}{12} = 2.$$

Solution C. In a polynomial, the term that dominates as $x \to -\infty$ is the term with the highest power of x. In the top polynomial, this term is $4x^3$; in the bottom polynomial, this term is $2x^3$. So,

$$\lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{4x^3}{2x^3} = \lim_{x\to-\infty} 2 = 2.$$

3.4.22. *Find* $\lim_{x\to\infty} \cos(x)$ *or show that the limit does not exist.*

Solution. As x increases, the values of $\cos(x)$ never stop oscillating between 1 and -1, so the function cannot approach any single value. Therefore, the limit does not exist.