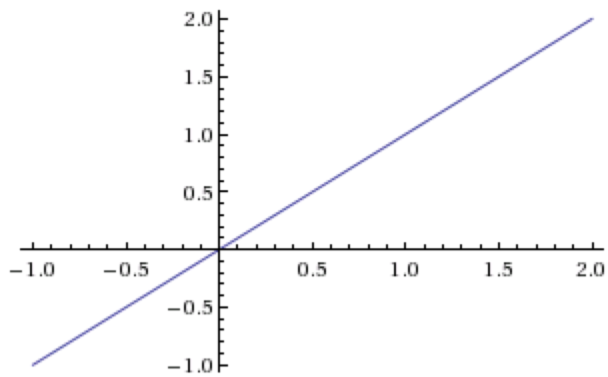
Math 2, Winter 2016

DAILY HOMEWORK #6 — SOLUTIONS

3.1.12. (a) *Sketch the graph of a function on* $[-1, 2]$ *that has an absolute maximum but no local maximum.*

Solution. Many functions will work. Here is one:

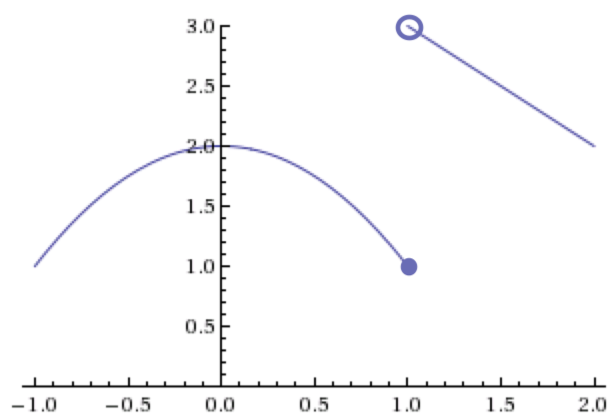


$y = x$ on $[-1, 2]$

This function has an absolute maximum at $x = 2$, but it has no local maximum: our convention is that a function does not have a local maximum at the endpoints of its domain.

(b) *Sketch the graph of a function on* $[-1, 2]$ *that has a local maximum but no absolute maximum.*

Solution. Any such function must have a discontinuity, as the extreme-value theorem says that every continuous function on a closed interval has an absolute maximum. Here is an example that works:



This function has a local maximum at $x = 0$; however, due to the discontinuity at $x = 1$, it does not have an absolute maximum.

3.4.25. *Find* $\lim_{x\to-\infty}(x^4+x^5)$ *or show that the limit does not exist.*

Solution A. $\lim_{x\to-\infty}(x^4+x^5) = \lim_{x\to-\infty} x^4(1+x)$; and, as $x\to-\infty$, we have $x^4\to\infty$ and $1+x\to-\infty$, so $\lim_{x\to-\infty} x^4(1+x) = -\infty$.

Solution B. As $x\to\pm\infty$, a polynomial function is dominated by its highest-degree term, which in this case is x^5; so $\lim_{x\to-\infty}(x^4-x^5) = \lim_{x\to-\infty} x^5 = -\infty$.

3.4.33. *Find the horizontal and vertical asymptotes of* $y = \dfrac{2x+1}{x-2}$.

Solution. To find the horizontal asymptotes, we take the limit as $x\to\infty$ or $x\to-\infty$:

$$\lim_{x\to\infty}\frac{2x+1}{x-2} = \lim_{x\to\infty}\frac{\frac{2x}{x}+\frac{1}{x}}{\frac{x}{x}-\frac{2}{x}} = \lim_{x\to\infty}\frac{2+\frac{1}{x}}{1-\frac{2}{x}} = 2,$$

and the same calculation shows that $\lim_{x\to-\infty}\dfrac{2x+1}{x-2} = 2$. Therefore, $y=2$ is the horizontal asymptote of the curve.

To find the vertical asymptotes, we find where the bottom of the fraction is zero. The only such number is $x=2$; and because 2 is not a zero of the top of the fraction, we know that $x=2$ is a vertical asymptote of the curve.

3.4.36. *Find the horizontal and vertical asymptotes of* $y = \dfrac{1+x^4}{x^2-x^4}$.

Solution. To find the horizontal asymptotes, we take the limit as $x\to\infty$ or $x\to-\infty$:

$$\lim_{x\to\infty}\frac{1+x^4}{x^2-x^4} = \lim_{x\to\infty}\frac{\frac{1}{x^4}+\frac{x^4}{x^4}}{\frac{x^2}{x^4}-\frac{x^4}{x^4}} = \lim_{x\to\infty}\frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1} = -1,$$

and the same calculation shows that $\lim_{x\to-\infty}\dfrac{1+x^4}{x^2-x^4} = -1$. Therefore, $y=-1$ is the horizontal asymptote of the curve.

To find the vertical asymptotes, we find where the bottom of the fraction is zero:

$$x^2-x^4=0 \quad\Rightarrow\quad x^2(1-x^2)=0 \quad\Rightarrow\quad x^2(1-x)(1+x)=0 \quad\Rightarrow\quad x=0,\ 1,\text{ or } -1.$$

Because none of these numbers is a zero of the top of the fraction (actually the top has no zeroes), we know that the curve has three vertical asymptotes: $x=-1$, $x=0$, and $x=1$.

3.5.9. *Use the guidelines of Section 3.5 to sketch the curve* $y = \dfrac{x}{x-1}$.

Solution. The domain of the function is $x\neq 1$. There is an x intercept at $x=0$, and the y intercept is $y=0$; this means the curve passes through the origin.

By taking the limit as $x\to\infty$ and the limit as $x\to-\infty$, we find that the horizontal asymptote is $y=1$, to the left and to the right. There is a vertical asymptote at $x=1$, because that is where the bottom of the fraction is zero; and

$$\lim_{x\to 1^-}\frac{x}{x-1}=-\infty \qquad \text{and} \qquad \lim_{x\to 1^+}\frac{x}{x-1}=\infty,$$

so the curve goes down toward $-\infty$ on the left of the vertical asymptote, and it goes up toward ∞ on the right of the vertical asymptote.

To find the intervals of increase or decrease, we take the derivative, using the quotient rule:

$$\frac{dy}{dx}=\frac{-1}{(x-1)^2}.$$

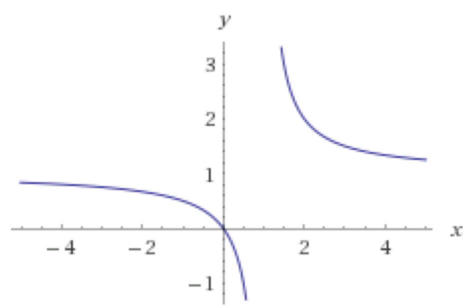
This is negative at every point in the domain, so in fact y is decreasing at every point in its domain. Consequently, there are no local extrema.

To find the intervals of concavity, we take the derivative again:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{-1}{(x-1)^2}=\frac{2}{(x-1)^3}.$$

This is negative on $(-\infty,1)$ and positive on $(1,\infty)$, so y is concave down on $(-\infty,1)$ and concave up on $(1,\infty)$. If 1 were in the domain if y, it would be an inflection point; but 1 is not in the domain.

Using this information, we can sketch the curve. It looks like this:



$$y=\frac{x}{x-1}$$

3.5.14. *Use the guidelines of Section 3.5 to sketch the curve* $y=\dfrac{x^2}{x^2+9}$.

Solution. The domain of the function is all numbers except the numbers where $x^2+9=0$. But x^2+9 does not have any real zeroes, so the domain of the function is all real numbers. There is an x intercept at $x=0$, and the y intercept is $y=0$; this means the curve passes through the origin.

By taking the limit as $x\to\infty$ and the limit as $x\to-\infty$, we find that the horizontal asymptote is $y=1$, to the left and to the right. Because the bottom of the fraction is never zero, there are no vertical asymptotes.

To find the intervals of increase or decrease, we take the derivative, using the quotient rule,

$$\frac{dy}{dx} = \frac{2x(x^2+9) - x^2(2x)}{(x^2+9)^2} = \frac{18x}{(x^2+9)^2}.$$

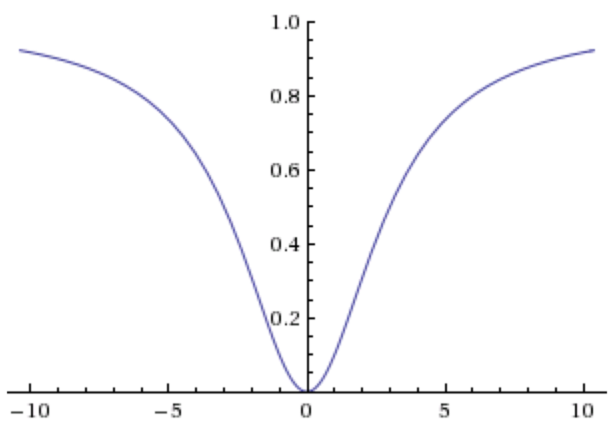
The derivative is never undefined, and it is zero when $x = 0$; the derivative is negative when $x < 0$ and positive when $x > 0$. Thus, y is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, and by the first-derivative test it has a local minimum at $x = 0$.

To find the intervals of concavity, we take the derivative again, now using the quotient rule and the chain rule:

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{18x}{(x^2+9)^2} &= \frac{18(x^2+9)^2 - (18x)(2x)2(x^2+9)}{(x^2+9)^4} \\ &= \frac{18(x^2+9) - 72x^2}{(x^2+9)^3} \\ &= \frac{54(3-x^2)}{(x^2+9)^3}.\end{aligned}$$

The second derivative is never undefined, and it is zero when $x = \pm\sqrt{3}$; by plugging in points, we find that $\frac{d^2y}{dx^2}$ is negative when $x < -\sqrt{3}$, positive when $-\sqrt{3} < x < \sqrt{3}$, and negative again when $x > \sqrt{3}$. Thus, y is concave down on $\left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)$, and y is concave up on $\left(-\sqrt{3}, \sqrt{3}\right)$. The inflection points occur when the concavity changes from down to up, or from up to down, so the two inflection points are at $x = \pm\sqrt{3} \approx \pm 1.73$.

Using this information, we can sketch the curve. It looks like this:



$$y = \frac{x^2}{x^2+9}$$