Math 2, Winter 2016

Daily Homework #8 — Solutions

3.7.15. *If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.*

Solution. Let s be the length of the side of the square base; let h be the height of the box; let V be the volume of the box. Then $V = hs^2$. We want to maximize V, so we need to write h in terms of s (or s in terms of h).

The surface area of the box is

$$(\text{area of base}) + 4 \cdot (\text{area of one side}) = s^2 + 4hs,$$

so $1200 = s^2 + 4hs$. We can solve this equation for h in terms of s:

$$4hs = 1200 - s^2 \quad \Rightarrow \quad h = \frac{1200 - s^2}{4s}.$$

Plugging this into the expression for V, we get $V = \dfrac{1200 - s^2}{4s} \cdot s^2 = 300\,s - \frac{1}{4}s^3$. Therefore,

$$V = 300\,s - \tfrac{1}{4}s^3,$$

and this is V as a function of one variable, s. What is its domain? The volume cannot be negative, which means $300\,s - \frac{1}{4}s^3 \geq 0$; solving this inequality for s yields $s \leq -20\sqrt{3}$ or $0 \leq s \leq 20\sqrt{3}$. But $s \geq 0$ rules out the possibility of $s \leq -20\sqrt{3}$, so the condition is $0 \leq s \leq 20\sqrt{3}$. Therefore, the domain is $\left[0, 20\sqrt{3}\right]$. (There are other ways of getting the righthand endpoint of this domain, or other reasonable choices of righthand endpoint.)

Now we need to find the absolute maximum of $V = 300\,s - \frac{1}{4}s^3$ on this domain. The derivative is $\frac{dV}{ds} = 300 - \frac{3}{4}s^2$, and this is never undefined, so we can find the critical numbers by setting it to zero:

$$300 - \tfrac{3}{4}s^2 = 0 \quad \Rightarrow \quad s^2 = 400 \quad \Rightarrow \quad s = \pm 20,$$

but -20 is not in the domain, so the only critical number is $s = 20$.

Now we can use the "closed-interval method" to determine the absolute maximum of V. We test the critical number and the endpoints of the domain:

$$V(0) = 0; \qquad V(20) = 300 \cdot 20 - \tfrac{1}{4}20^3 = 4000; \qquad V\left(20\sqrt{3}\right) = 0.$$

The highest of these is $V(20) = 4000$, so the absolute maximum is 4000. Therefore, the largest possible volume is 4000 cm^3. (This is achieved when the base side length is $s = 20$ cm and the height is $h = \dfrac{1200 - 20^2}{4 \cdot 20} = 10$ cm.)