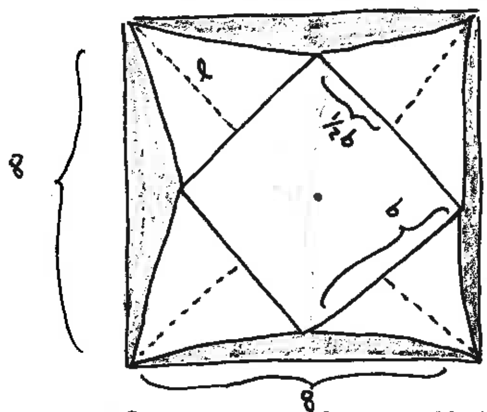
Challenging Optimization Problems

You will have all class to work on these problems with those around you. Whatever you do not finish during class will be your homework assignment for over the weekend.

1) Given an 8 inch by 8 inch piece of paper, build a pyramid with a square base that has the greatest volume.

$$V = \frac{1}{3}b^2h$$

where b is the length of the base of the pyramid and h is the height of the pyramid.

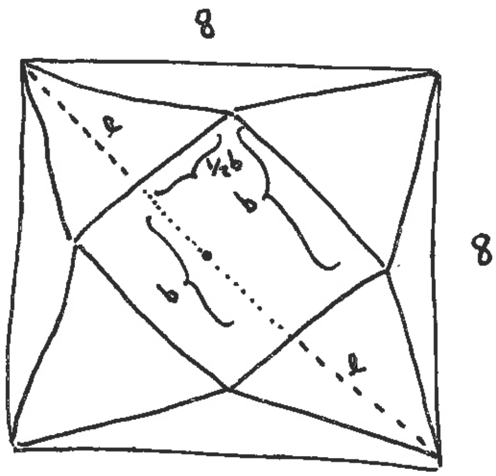


Questions to ask yourself along the way:

- What is the length of the diagonal of my paper?
- How do the lengths of b and l relate?
- When you fold the pyramid up, how do you find the height of the pyramid?

2) You are building a glass fish tank that will hold 72 cubic feet of water. You want its base and sides to be rectangular and the top to be open. You want to construct the tank so that its width is 5 feet, but the length and depth are variable. Building materials for the tank cost 10 dollars per square foot for the base and 5 dollars per square foot for the sides. What are the dimensions of the least expensive tank and what is the cost of the least expensive tank?

3) A ladder must reach over a fence that is 8 feet high with a wall that is 1 foot behind the fence. What is the length of the shortest ladder you can use?
Hint: Draw a picture and look for similar triangles - triangles that have the same angles, but may be a different size. And remember that similar triangles will satisfy $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$.



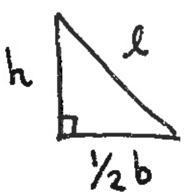
The sides of the paper are 8 inches long each.

Therefore, by the Pythagorean Theorem, the lenghth of the diagonal is
$c^2 = 8^2 + 8^2 \Rightarrow c = \sqrt{8^2+8^2} = 8\sqrt{2}$.

An important observation is that the diagonal c is also equal to $\ell + b + \ell$!
Therefore, we have $\boxed{8\sqrt{2} = 2\ell + b}$.

Now, how do we figure out the height of the pyramid? This is where building a model is very useful, because when you fold up the sides...

the Δ side becomes the hypotenuse. for a triangle right down the middle of your pyramid.



And using the Pythagorean theorem,
$h^2 + (\frac{1}{2}b)^2 = \ell^2$ and solve for h to get
$h = \sqrt{-\frac{1}{4}b^2 + \ell^2} = \boxed{\sqrt{\ell^2 - \frac{1}{4}b^2} = h}$

You want to maximize Volume. And Volume$_\Delta$ = $\frac{1}{3}b^2h$. Writing everything in terms of b...

$8\sqrt{2} = 2\ell + b \Rightarrow \ell = \frac{1}{2}(8\sqrt{2} - b)$

$h = \sqrt{\ell^2 - \frac{1}{4}b^2} = \sqrt{\frac{1}{4}(8\sqrt{2}-b)^2 - \frac{1}{4}b^2} = \sqrt{\frac{1}{4}(b^2 - 16\sqrt{2}b + 128) - \frac{1}{4}b^2}$

$\boxed{h = \sqrt{32 - 4\sqrt{2}\,b}}$

So. $V(b) = \frac{1}{3}b^2\sqrt{32 - 4\sqrt{2}\,b}$

Now, take the derivative and set it to zero to find your critical points.

$$V(b) = \frac{1}{3} b^2 \sqrt{32 - 4\sqrt{2}\, b}$$

$$\Rightarrow V'(b) = \frac{2}{3} b \sqrt{32 - 4\sqrt{2}\, b} + \frac{1}{3} b^2 \left(\frac{-4\sqrt{2}}{2\sqrt{32 - 4\sqrt{2}\, b}} \right) = 0$$

(multiply entire by $\sqrt{32 - 4\sqrt{2}\, b}$)

$$\frac{2}{3} b \left(32 - 4\sqrt{2}\, b\right) + \frac{1}{3}\left(-2\sqrt{2}\right) b^2 = 0$$

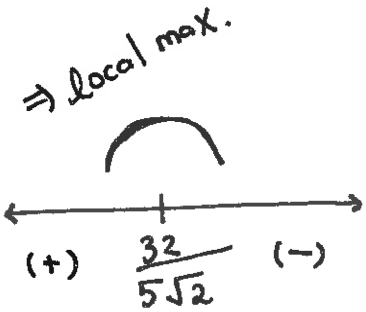
$$\frac{64}{3} b - \frac{8\sqrt{2}}{3} b^2 - \frac{2\sqrt{2}}{3} b^2 = 0$$

$$64 b - 10\sqrt{2}\, b^2 = 0 \qquad b \neq 0 \text{ so}$$

$$64 - 10\sqrt{2}\, b = 0$$

$$b = \frac{64}{10\sqrt{2}} = \frac{32}{5\sqrt{2}} = \frac{16\sqrt{2}}{5}$$

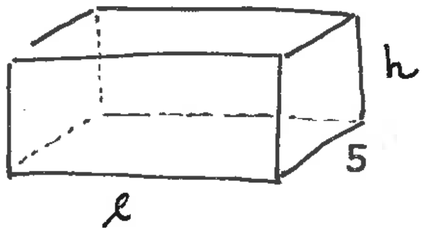
$$\approx 4.525$$



And what is the volume?

$$V\left(\tfrac{32}{5\sqrt{2}}\right) = \frac{1}{3}\left(\frac{32}{5\sqrt{2}}\right)^2 \sqrt{32 - 4\sqrt{2}\left(\frac{32}{5\sqrt{2}}\right)}$$

$$= \approx 17.2703 \ \text{in}^3$$



Volume of fish tank:

$$V = \ell \cdot w \cdot h$$

$$\Rightarrow \boxed{72 = \ell \cdot 5 \cdot h}$$

Cost of fish tank:

$$\text{Cost} = \underbrace{10(5\ell)}_{\text{bottom}} + \underbrace{5(2\cdot 5h + 2\ell h)}_{\text{sides.}}$$

$$C = 50\ell + 50h + 10\ell h.$$

Write in terms of one variable.

$$\boxed{h = \frac{72}{5\ell}}$$

$$C(\ell) = 50\ell + 50\left(\frac{72}{5\ell}\right) + 10\ell\left(\frac{72}{5\ell}\right)$$

$$C(\ell) = 50\ell + 720/\ell + 144$$

To find critical points, take the derivative and set it to zero.

$$C'(\ell) = 50 - 720/\ell^2.$$

$$\Rightarrow 0 = 50 - \frac{720}{\ell^2}$$

$$\ell^2 = 14.4$$

$$\boxed{\begin{array}{l} \Rightarrow \ell = 3.795 \\ \Rightarrow h = 3.795 \end{array}}$$

dimensions.

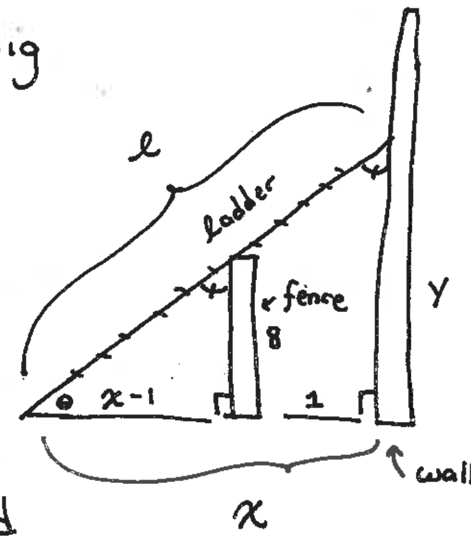
c': (−) 3.795 (+)

And the least expensive tank costs $\boxed{\$523.473}$

By the Pythagorean theorem, the big triangle satisfies:

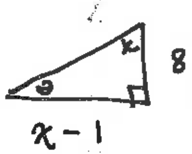
$$\ell^2 = x^2 + y^2.$$

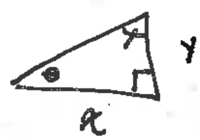
$$\Rightarrow \quad \ell = \sqrt{\underbrace{x^2 + y^2}_{\text{minimize the inside.}}}$$



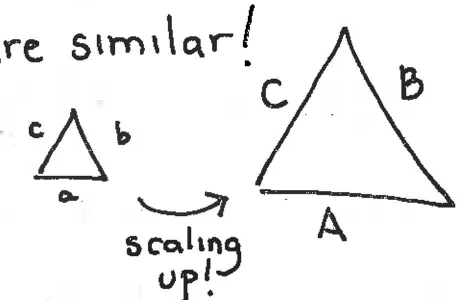
we must minimize $L = x^2 + y^2$.

Now, to find our restriction, we need a clever observation. The two triangles



and


are similar!



Therefore,

$$\frac{x-1}{x} = \frac{8}{y}$$ } because we have just scaled up our Δ.

$$\star \quad \frac{a}{A} = \frac{b}{B} = \frac{c}{C}.$$

$$\Rightarrow \boxed{y = \frac{8x}{x-1}}$$

$$L = x^2 + y^2$$

So, writing L in one variable $L = x^2 + \left(\frac{8x}{x-1}\right)^2 = x^2 + \frac{64x^2}{(x-1)^2}$.

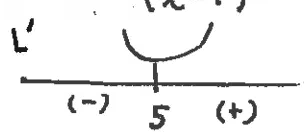
To find critical points, take derivative and set to zero.

$$L'(x) = 2x + \frac{128x(x-1)^2 - 2(x-1)(64x^2)}{(x-1)^4} = \ldots = 2x - \frac{128x}{(x-1)^3}.$$

$$0 = x\left(2 - \frac{128}{(x-1)^3}\right)$$

$x = 0$

$$2 = \frac{128}{(x-1)^3} \Rightarrow 64 = (x-1)^3 \qquad 4^3 = 64$$



$$\Rightarrow 4 = x - 1$$

$$\Rightarrow \boxed{x = 5} \text{ and } \boxed{y = \frac{8(5)}{5-1} = 10.}$$

$$\ell = \sqrt{x^2 + y^2} = \sqrt{5^2 + 10^2} = \sqrt{125} \approx \boxed{11.180 \text{ ft} \leftarrow \text{shortest ladder}}$$