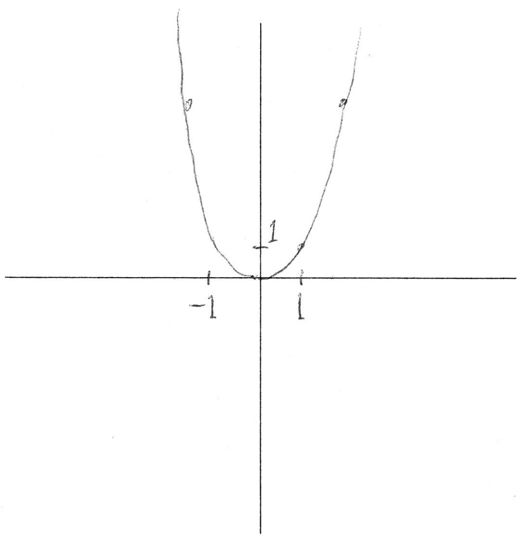
Weekly Homework #1
Due Friday, January 8

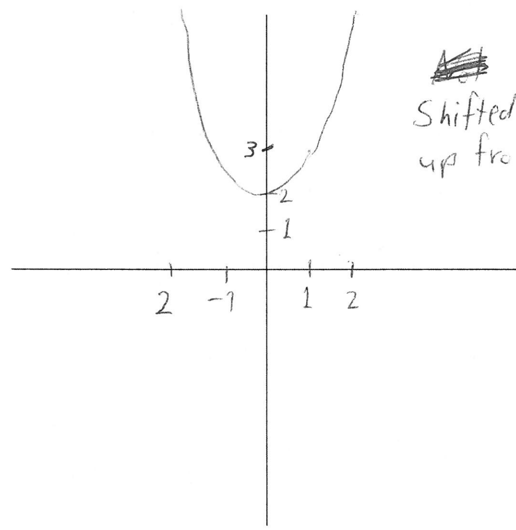
SOLUTIONS

1. Sketch the graph of the given function. Remember to label the axes with tick marks.

(a) $f(x) = x^2$

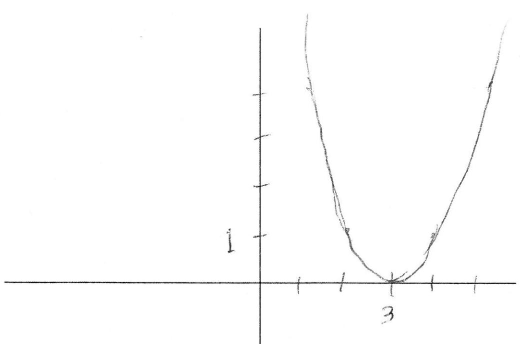


(b) $f(x) = x^2 + 2$



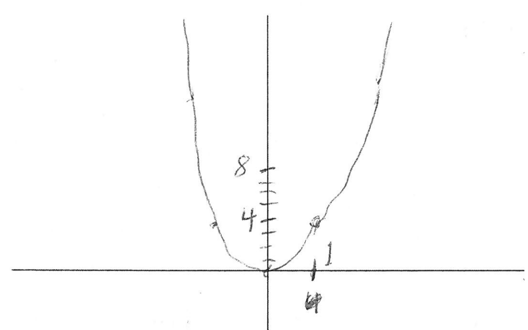
Shifted 2 up from x^2.

(c) $f(x) = (x - 3)^2$



Shifted 3 to the right from x^2

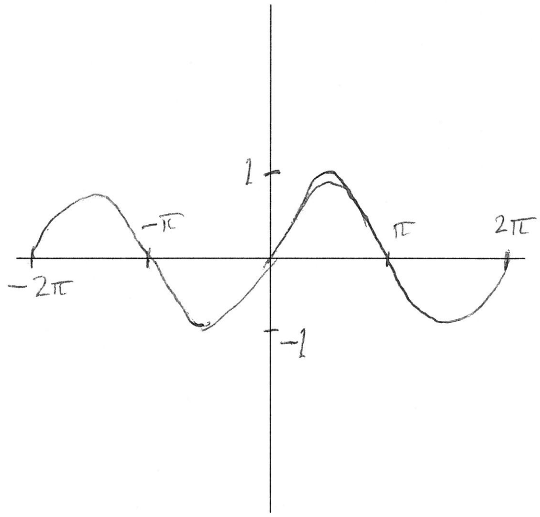
(d) $f(x) = 4x^2$



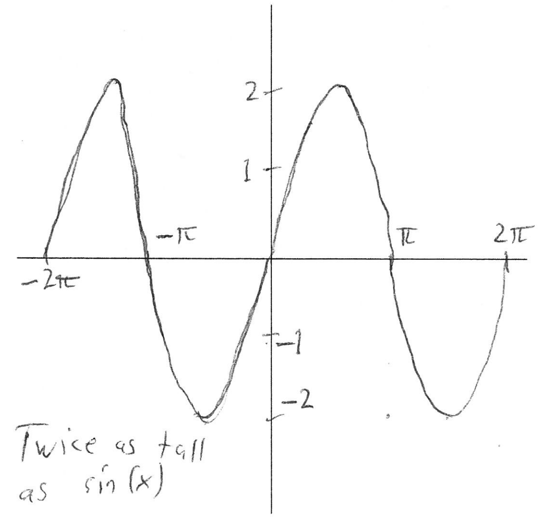
4 times as tall as x^2 — note the different scale on the vertical axis.

SOLUTIONS

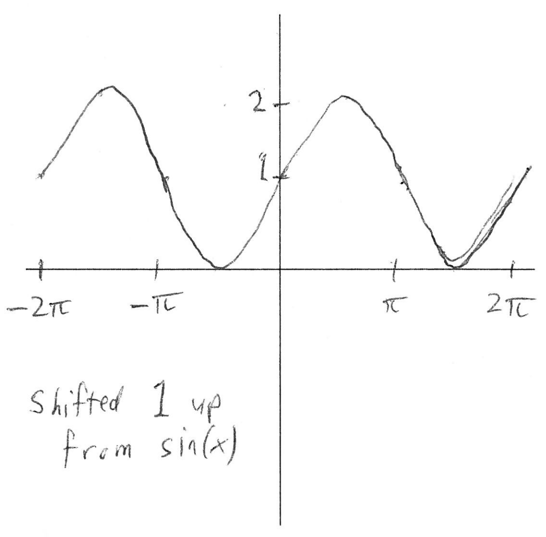
(e) $f(x) = \sin(x)$



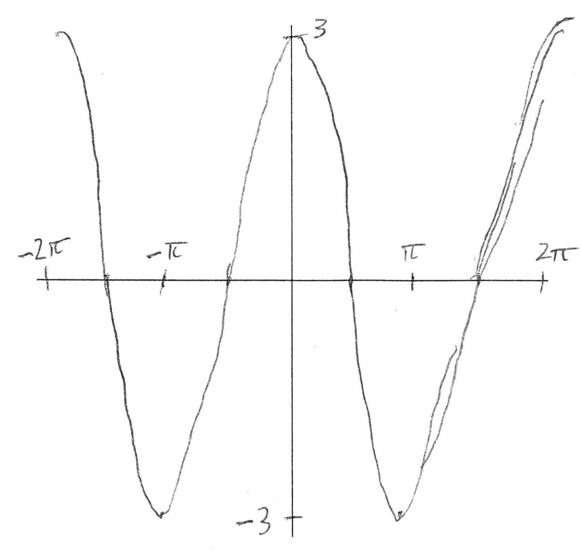
(f) $f(x) = 2\sin(x)$



(g) $f(x) = \sin(x) + 1$



(h) $f(x) = 3\cos(x)$



SOLUTIONS

2. Find the equation of the tangent line to the given function at the specified point.

(a) $f(x) = 4x^3 + x + 1$ at $x = 1$.

The slope of the tangent line is $f'(1)$.

$f'(x) = 12x^2 + 1$.

$f'(1) = 12 \cdot 1^2 + 1 = 13$.

A point on the tangent line is $(1, f(1)) = (1, 6)$.

So, using point-slope form, the tangent line is $\boxed{y = 13(x-1) + 6}$

(b) $f(x) = \dfrac{2x}{x^2+1}$ at $x = 1$.

The slope is $f'(1)$.

Quotient rule: $f'(x) = \dfrac{(x^2+1)\cdot(2x)' - 2x\cdot(x^2+1)'}{(x^2+1)^2}$

$= \dfrac{(x^2+1)\cdot 2 - 2x\cdot 2x}{(x^2+1)^2}$

$f'(x) = \dfrac{-2x^2+2}{(x^2+1)^2}$.

$f'(1) = \dfrac{-2(1^2)+2}{(1^2+1)^2} = 0$. So the slope is 0 — a horizontal line through $f(1) = \dfrac{2 \cdot 1}{1^2+1} = 1$. Equation: $\boxed{y=1}$.

3. Find the derivative of the given function.

(a) $f(x) = \sin^2(x)$

Solution 1. Chain rule.

$f(x) = (\sin(x))^2$.

$\frac{d}{du} u^2 = 2u$

$\frac{d}{dx} \sin(x) = \cos(x)$.

So $\frac{d}{dx}(\sin(x))^2 = \boxed{2\sin(x)\cos(x)}$.

Solution 2. Product rule.

$f(x) = \sin(x)\sin(x)$

$f'(x) = (\sin(x))'\sin(x) + \sin(x)(\sin(x))'$

$= \cos(x)\sin(x) + \sin(x)\cos(x)$

$= \boxed{2\sin(x)\cos(x)}$.

(b) $f(x) = \cos(3x^2)$

Chain rule.

$\frac{d}{du}\cos(u) = -\sin(u)$; $\frac{d}{dx}(3x^2) = 6x$; so $\frac{d}{dx}\cos(3x^2)$

$= \boxed{-\sin(3x^2)\cdot 6x}$

SOLUTIONS

4. Use L'Hôpital's rule to find $\lim_{x\to 0} \frac{\tan(x)}{x}$.

$\frac{\tan(x)}{x} \to \frac{0}{0}$, so $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{(\tan(x))'}{(x)'}$

$= \lim_{x\to 0} \frac{\sec^2(x)}{1} = \sec^2(0) = \boxed{1}$.

5. Let $g(x) = x^3 e^x$. Find $g''(x)$.

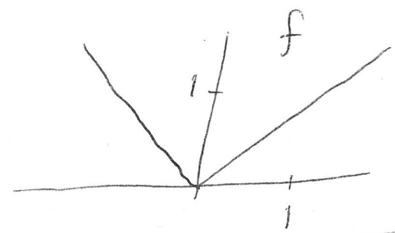
Product rule: $g'(x) = (x^3)' e^x + x^3 (e^x)'$
$= 3x^2 e^x + x^3 e^x$.

Product rule again: $g''(x) = (3x^2 e^x)' + (x^3 e^x)'$
$= [(3x^2)' e^x + 3x^2 (e^x)'] + [(x^3)' e^x + x^3 (e^x)'] = \boxed{6x e^x + 3x^2 e^x + 3x^2 e^x + x^3 e^x}$
$\boxed{= (6x + 6x^2 + x^3) e^x}$

6. Is the given function continuous? If not, at which value(s) of x is it discontinuous?

(a) $f(x) = |x|$

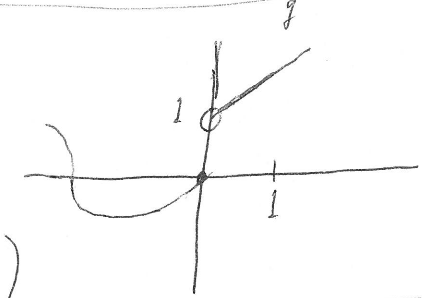
YES. It has a corner at 0, but that makes it non-differentiable, not discontinuous.



(b) $g(x) = \begin{cases} \sin(x) & \text{if } x \le 0 \\ x+1 & \text{if } x > 0 \end{cases}$

g is discontinuous at $x = 0$, but continuous everywhere else.

$\left(\lim_{x\to 0^-} g(x) = \sin(0) = 0, \text{ but } \lim_{x\to 0^+} g(x) = 0 + 1 = 1 \right)$



(c) $h(x) = \begin{cases} 5x-4 & \text{if } x < 1 \\ 1/x & \text{if } x \ge 1 \end{cases}$

YES. As $5x-4$ is continuous for $x < 1$, and $\frac{1}{x}$ is continuous for $x \ge 1$ (though $\frac{1}{x}$ is undefined at $x=0$), we just need to check that h is continuous at 1:

$\lim_{x\to 1^-} 5x-4 = 1$, and $\lim_{x\to 1^+} \frac{1}{x} = 1$, so it is.