Math 2, Winter 2016

Weekly Homework #8 — Solutions

9.1.9. *A population is modeled by the differential equation*

$$\frac{dP}{dt} = 1.2\,P\left(1 - \frac{P}{4200}\right).$$

(a) *For what values of P is the population increasing?*

Solution. P is increasing when $\frac{dP}{dt} > 0$. This means $P\left(1 - \frac{P}{4200}\right) > 0$, which occurs precisely when $0 < P < 4200$.

(b) *For what values of P is the population decreasing?*

Solution. P is decreasing when $\frac{dP}{dt} < 0$. This means $P\left(1 - \frac{P}{4200}\right) < 0$, which occurs precisely when $P < 0$ or $P > 4200$. (If P represents a real-life population, then P cannot be negative, so the answer would just be $P > 4200$.)

(c) *What are the equilibrium solutions?*

Solution. The equilibrium solutions are when $\frac{dP}{dt} = 0$. This means $P\left(1 - \frac{P}{4200}\right) = 0$, which happens precisely when $P = 0$ or $P = 4200$. Thus the equilibrium solutions are $P = 0$ and $P = 4200$.

9.1.13. *Match the differential equations with the solution graphs. Give reasons for your choices.*

Solution. (a) When the point (x, y) gets farther from the origin, the slope $y' = 1+x^2+y^2$ gets higher. Also, $y' = 1 + x^2 + y^2$ is always positive, so the graph is always increasing. The graph with these properties is III.

(b) Notice that $e^{-x^2-y^2}$ is always positive; so $y' = xe^{-x^2-y^2}$ is negative for $x < 0$ and positive for $x > 0$ — meaning that the graph should decrease for $x < 0$ and increase for $x > 0$. The graph with these properties is I.

(c) When the point (x, y) gets farther from the origin, the quantity $1+e^{x^2+y^2}$ gets higher, so the slope $y' = \frac{1}{1 + e^{x^2+y^2}}$ gets lower — so the slope gets lower as (x, y) gets farther from the origin. But also, $y' = \frac{1}{1 + e^{x^2+y^2}}$ is always positive, so the graph is always increasing. The graph with these properties is IV.

(d) The graph is II because it's the only one left. But we can also see why the answer is II because the graph is the constant function $y = 0$. We can check that $y = 0$ is not a solution to the differential equations in (a), (b), and (c); but it is a solution to (d).

9.3.46. *The air in a room with volume* 180 m^3 *contains* 0.15% *carbon dioxide initially. Fresher air with only* 0.05% *carbon dioxide flows into the room at a rate of* 2 m^3/min *and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?*

Solution. Let $y(t)$ be the total volume of carbon dioxide in the room.

- Fresh air enters the room at rate 2 m^3/min, and 0.05% of this air is carbon dioxide, so the rate at which carbon dioxide enters the room is $2 \cdot 0.0005 = 0.001$ m^3/min.
- The air mixture leaves the room at rate 2 m^3/min, and the concentration of carbon dioxide in this air is $y/180$, so the rate at which carbon dioxide leaves the room is $2 \cdot y/180 = y/90$ (this is in m^3/min).

Thus, the differential equation that governs the system is

$$\frac{dy}{dt} = 0.001 - \frac{y}{90}.$$

This is a separable differential equation, so we can solve it as follows:

$$\int \frac{dy}{0.001 - \frac{y}{90}} = \int dt.$$

The integral on the right is $t + K$. To do the integral on the left, we substitute $u = 0.001 - \frac{y}{90}$ and $du = -\frac{1}{90} dy$:

$$\begin{aligned}\int \frac{dy}{0.001 - \frac{y}{90}} = -90 \int \frac{1}{0.001 - \frac{y}{90}} \cdot \left(-\tfrac{1}{90}\right) dy &= -90 \int \frac{1}{u}\, du \\ &= -90 \ln|u| = -90 \ln\left|0.001 - \frac{y}{90}\right|.\end{aligned}$$

So we get

$$\begin{aligned} -90 \ln\left|0.001 - \frac{y}{90}\right| &= t + K; \\ \ln\left|0.001 - \frac{y}{90}\right| &= -\frac{t}{90} + C; && \text{(setting } C = -K/90\text{)} \\ \left|0.001 - \frac{y}{90}\right| &= e^{-(t/90)+C} \\ 0.001 - \frac{y}{90} &= \pm e^C e^{-t/90}; \\ \frac{y}{90} &= 0.001 - Be^{-t/90}; && \text{(setting } B = \pm e^C\text{)} \\ y &= 0.09 + Ae^{-t/90}. && \text{(setting } A = -90B\text{)} \end{aligned}$$

Therefore, the general solution is $y = 0.09 - Ae^{-t/90}$.

Now we need to use the initial condition: the room contains 0.15% carbon dioxide at time $t = 0$, so $y(0) = 180 \cdot 0.0015 = 0.27\ \mathrm{m}^3$. Then

$$0.27 = 0.09 + Ae^0 \quad \Rightarrow \quad A = 0.18;$$

so the particular solution that describes this system is

$$y(t) = 0.09 + 0.18\, e^{-t/90}.$$

But $y(t)$ is the *volume* of carbon dioxide; what we need is the *percentage* of carbon dioxide. If the percentage of carbon dioxide is $p(t)$, then

$$p(t) = 100\% \cdot \frac{y(t)}{180} = 0.05\% + \left(0.1\% \cdot e^{-t/90}\right).$$

To find out what happens in the long run, we can take the limit as $t \to \infty$:

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} 0.05\% + \left(0.1\% \cdot e^{-t/90}\right) = 0.05\%.$$

So the percentage of carbon dioxide will approach 0.05%. We didn't really need math to know this — of course this is going to happen when air with 0.05% carbon dioxide flows into the room.

9.4.4. *Suppose a population $P(t)$ satisfies*

$$\frac{dP}{dt} = 0.4\,P - 0.001\,P^2, \qquad P(0) = 50,$$

where t is measured in years.

(a) *What is the carrying capacity?*

Solution. We can rewrite the differential equation as

$$\frac{dP}{dt} = 0.4\,P\left(1 - \frac{P}{400}\right),$$

which represents logistic growth with carrying capacity 400.

(b) *What is P'(0)?*

Solution. Substitute the initial condition $P(0) = 50$ into the differential equation:

$$\frac{dP}{dt} = 0.4 \cdot 50 - 0.001 \cdot 50^2 = 17.5;$$

so $P'(0) = 17.5$.

(c) *When will the population reach 50% of the carrying capacity?*

Solution. The general solution to the logistic growth equation, with proportionality constant $k = 0.4$ and carrying capacity $M = 400$, is

$$P(t) = \frac{400}{1 + Ae^{-0.4\,t}}.$$

To find A, substitute $P(0) = 50$:

$$50 = \frac{400}{1 + Ae^0} \quad \Rightarrow \quad 50(1 + A) = 400 \quad \Rightarrow \quad A = 7.$$

Then the particular solution is

$$P(t) = \frac{400}{1 + 7e^{-0.4\,t}}.$$

Now, 50% of the carrying capacity is 200, so we set $P(t) = 200$:

$$\begin{aligned} \frac{400}{1 + 7e^{-0.4\,t}} &= 200; \\ 1 + 7e^{-0.4\,t} &= 2; \\ e^{-0.4\,t} &= 1/7; \\ 0.4\,t &= \ln(7); \\ t &= \frac{\ln(7)}{0.4} \approx 4.86 \text{ years.} \end{aligned}$$

6.5.10. *A sample of tritium-3 decayed to* 94.5% *of its original amount after a year.*

(a) *What is the half-life of tritium-3?*

Solution. Let $m(t)$ be the mass remaining, with t measured in years. Then $m(t)$ is an exponential decay function:

$$m(t) = m_0 e^{kt},$$

where m_0 is the original mass and k is a negative constant. To find k, we set $m(1) = 0.945\,m_0$ and solve for k:

$$\begin{aligned} m_0 e^k &= 0.945\,m_0; \\ e^k &= 0.945; \\ k &= \ln(0.945). \end{aligned}$$

Then

$$m(t) = m_0 e^{\ln(0.945)\cdot t};$$

and if you want you can write this as

$$m(t) = m_0 \cdot 0.945^t.$$

To find the half-life, we set $m(t) = \frac{1}{2}m_0$ and solve for t:

$$\begin{aligned} m_0 e^{\ln(0.945)\cdot t} &= \tfrac{1}{2}m_0; \\ e^{\ln(0.945)\cdot t} &= \tfrac{1}{2}; \\ \ln(0.945)\cdot t &= \ln(0.5); \\ t &= \frac{\ln(0.5)}{\ln(0.945)} \approx 12.3 \text{ years.} \end{aligned}$$

(b) *How long would it take the sample to decay to* 20% *of its original amount?*

Solution. We set $m(t) = 0.2\, m_0$ and solve for t:

$$\begin{aligned} m_0 e^{\ln(0.945)\cdot t} &= 0.2\, m_0; \\ e^{\ln(0.945)\cdot t} &= 0.2; \\ \ln(0.945)\cdot t &= \ln(0.2); \\ t &= \frac{\ln(0.2)}{\ln(0.945)} \approx 28.5 \text{ years.} \end{aligned}$$