Math 3, Midterm 1 Solutions

October 21, 2009

For the multiple choice questions, we omit the choices and just calculate the answer.

1. For the function

$$f(x) = \frac{x-1}{x^2} = x^{-1} - x^{-2},$$

what is its domain?

Solution. This function is defined everywhere except $x = 0$; therefore, the domain is $(-\infty, 0) \cup (0, \infty)$, which is choice **B**.

2. For the same function as question 1, what are its vertical and horizontal asymptotes?

Solution. The line $x = 0$ is evidently a vertical asymptote, because $f(x)$ is not defined there, and $\lim_{x\to 0} f(x) = -\infty$. For horizontal asymptotes, we analyze $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, which are both equal to 0. Therefore $y = 0$ is a horizontal asymptote, and so choice **A** is correct.

3. A weight that is suspended by a spring of length 1 meter bobs up and down with its height above the oor at time t seconds given by the formula

$$h(t) = 2 + \frac{1}{\pi}\sin(\pi t).$$

Note that the function $h(t)$ has the derivative

$$h'(t) = \cos(\pi t).$$

At $t = 1/2$ seconds the object is

Solution. **C**, at its highest point. Indeed, $h(t)$ is largest (and hence the object is highest) when $\sin \pi t = 1$, which occurs precisely when $t = 1/2, 5/2, 9/2, \ldots$.

4. For the same situation as problem 3, but at time $t = 1$ second, the object is

Solution. **B**, moving its fastest downward. A graph of the function h will show that the object is moving downwards at time $t = 1$; also, the fact that $h'(1) =$

$\cos(\pi) = -1$ indicates that the object is moving downwards. As a matter of fact, the object is moving its fastest downwards, because -1 is the most negative value that $h'(t)$ takes.

5. The function $f(x) = \cos x + \sin x + 1$ is

Solution. **C**, neither even nor odd. $f(x)$ certainly is not odd, because $f(0) = 2 \neq -f(0)$. Also, $f(x)$ is not even, since $f(\pi/2) = 2$, while $f(-\pi/2) = 0$, which are not equal to each other.

6. Let $f(x) = \ln(2x - 1)$ and $g(x) = e^{2x}$. Then $g(f(x))$ is

Solution. **D**, $4x^2 - 4x + 1$. For readability, we will write $\exp n$ for e^n. We have

$$g(f(x)) = \exp(2\ln(2x - 1)).$$

We can factor $x^2 + 2x + 1 = (x + 1)^2$, so
Since $2\ln(2x - 1) = \ln(2x - 1)^2$, we have

$$g(f(x)) = \exp(\ln(2x - 1)^2) = (2x - 1)^2 = 4x^2 - 4x + 1.$$

7. Solve for x, expressing your answer in terms of the natural logarithm function:

$$8^{x+2} = 5^{-x}.$$

Solution. Take the natural log of both sides to obtain

$$\ln 8^{x+2} = \ln 5^{-x} \Leftrightarrow (x + 2)\ln 8 = -x \ln 5$$

Now we solve for x in the usual way:

$$x(\ln 8 + \ln 5) = -2\ln 8 \Rightarrow x = \frac{-2\ln 8}{\ln 8 + \ln 5}$$

We can simplify the denominator to $\ln 40$, using $\ln a + \ln b = \ln ab$. Therefore, the correct choice is **C**.

8. Find $f'(x)$ when $f(x) = 3x^2 - 5x + \sqrt{3}$.

Solution. Use the power rule to find $f'(x) = 6x - 5$, which is choice **A**.

9. Consider the function $f(x) = -x + 1$, for $x \leq 1$, and x^2, for $x > 1$. Which of the following statements is true?

Solution. **C**, that $x = 1$ is a discontinuity of f which is not removable, is correct. Indeed, $\lim_{x\to 1^-} f(x) = 0$, while $\lim_{x\to 1^+} f(x) = 1$. Therefore, no definition of $f(x)$ at $x = 1$ will make $f(x)$ continuous, since $\lim_{x\to 1} f(x)$ does not even exist.

10. For the limit

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4},$$

which of the following is true?

Solution. **D**, the limit is 1/4, is correct. Notice that the numerator and denominator of the function are both 0 at $x = 2$, so we factor both of them and notice that the term $x - 2$ cancels:

$$\frac{x^2-3x+2}{x^2-4} = \frac{(x-2)(x-1)}{(x-2)(x+2)} = \frac{x-1}{x+2}$$

where the rightmost equality holds whenever $x \neq 2$. Therefore, the limit of the original fraction as $x \to 2$ is $(2-1)/(2+2) = 1/4$.

11. Find the derivative of the function: $f(x) = \sin(3x+1) + \cos(2x-3)$.

Solution. Use the chain rule, and the formulas for the derivative of sin and cos, to obtain

$$f'(x) = 3\cos(3x+1) - 2\sin(2x-3)$$

which is choice **E**, none of the above.

12. Find $f'(x)$ if $f(x) = \tan \frac{x+1}{x-1}$.

Solution. We use the chain rule to find

$$f'(x) = \sec^2 \frac{x+1}{x-1} \cdot \left(\frac{x+1}{x-1}\right)'$$

The derivative of $(x+1)/(x-1)$ can be evaluated using the quotient rule:

$$\left(\frac{x+1}{x-1}\right)' = \frac{1\cdot(x-1) - 1\cdot(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

This gives choice **C**. We could have also calculated the derivative of $(x+1)/(x-1)$ by rewriting it as follows:

$$\frac{d}{dx}\frac{x+1}{x-1} = \frac{d}{dx}\left(1 + \frac{2}{x-1}\right) = \frac{-2}{(x-1)^2}$$

13. The function $f(x) = \sqrt{x}$ has the derivative $f'(x) = 1/(2\sqrt{x})$. Find the equation of the tangent line to to the graph of $y = f(x)$ at the point $(4, 2)$.

Solution. The tangent line in question passes through the point $(4, 2)$ and has slope equal to $f'(4)$. Therefore, the slope is $f'(4) = 1/4$, and the equation of the tangent line can be found using the point-slope form for a line:

$$y - 2 = 1/4(x - 4)$$

If we wanted to, we could write this in the form $y = x/4 + 1$.

14. Carefully work out the derivative of $f(x) = 2x^2 - 3x$ using the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Solution. Plug in the expression for $f(x)$ into the definition of $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$$

We expand out the terms in the numerator and simplify:

$$2(x+h)^2 - 3(x+h) - (2x^2 - 3x) = 2(x^2 + 2hx + h^2) - 3x - 3h - 2x^2 + 3x = 4hx + 2h^2 - 3h$$

Therefore, $f'(x)$ is equal to

$$f'(x) = \lim_{h \to 0} \frac{4hx + 2h^2 - 3h}{h} = \lim_{h \to 0} 4x - 3 + 2h = 4x - 3.$$

This is in agreement with what the power rule would give us.