Math 38 – Graph Theory | Nadia Lafrenière

Vertex Degrees and counting

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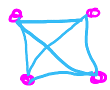
Today, we are doing a bit of combinatorics and will deduce some properties on the degrees, number of edges and number of vertices.

We already defined the degree of a vertex in a loopless graph to be the number of edges incident to it.
For a general graph, define the degree $d_G(v)$ of the vertex v to be the number of edges incident to it, with each loop counted twice.

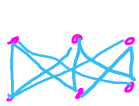
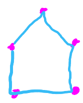
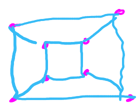
The maximum degree of a vertex is denoted $\Delta(G)$ and the minumum degree is denoted $\delta(G)$.
A graph is said to be regular if $\delta(G) = \Delta(G)$

The order of a graph $G=(V,E)$ is $|V|$, as the size of G is $|E|$.

Example



- K_n is a regular graph. Each vertex has degree $n-1$.
- $K_{m,n}$ is regular if and only if $m=n$. Then, the degree is always n.
- A connected regular graph that has the same order and size is a cycle.
- Hypercubes are regular graphs.

Counting and bijections

Proposition (degree-sum formula)
If $G=(V,E)$ is a graph, then $\sum_{v\in V} d_G(v) = 2|E|$

Proof
For each edge, there are two endpoints (maybe equal). If the endpoints are different, this edge contributes for 1 in the degree of two different vertices. If the edge is a loop, it adds 2 to the degree of the vertex it is incident to. So either way, every edge accounts for 2 in the total degree count.

Corollary
In any graph $G=(V,E)$, the average degree is $2|E|/|V|$, and

$$\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$$

Corollary

Every graph has an even number of vertices of odd degree.

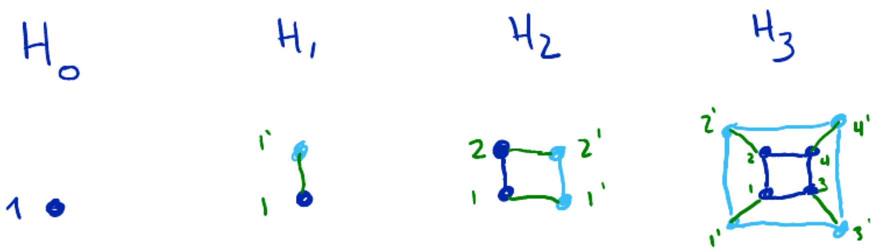
Corollary

A k-regular graph (i.e. a regular graph in which the degree of each vertex is k) has $k|V|/2$ edges.

Example: Hypercubes

The n-dimensional hypercube H_n is defined recursively as:

- H_0 is the simple graph with one vertex
- H_{n+1} is obtained by creating two copies of H_n and appending an edge between corresponding vertices in the two copies.



Proposition

H_n is regular, as each vertex has degree n.

Proof

The proof can be done by regular induction.

The base case is H_0, and it has no edge.

Induction hypothesis: The n-dimensional hypercube H_n is n-regular.

Induction step: The $(n+1)$-dimensional hypercube is made of two copies of H_n, and we add an edge between every pair of similar vertices in the two copies. This way, we add exactly one to the degree of each vertex from H_n, and that degree is, by induction hypothesis n.

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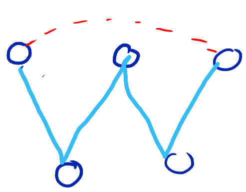
Proposition

If $k>0$, then a k-regular bipartite graph has the same number of vertices in its two independent sets.



Either not bipartite or not regular.

Proof

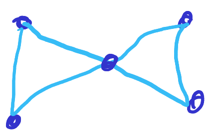
Since the graph is regular, all vertices have degree k. If there are m edges in total, the sum of the degrees for all the vertices in one independent set is m, as every edge has exactly one endpoint in that set. Since the graph is k-regular, there are m/k vertices in each set, so the order of both sets is the same.

Vertex-deletion and reconstruction conjecture

Is it possible to reconstruct a graph if you have only a list of its subgraphs? There is a long-standing, and still open conjecture saying it is, and so far we know it is almost always possible (that being understood in a probabilistic sense).

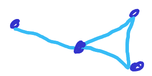
For a graph G, a vertex-deleted subgraph is an induced subgraph $G-v$ obtained by deleting a single vertex v.

Example



has vertex-deleted subgraphs

1 × 

4 × 

Proposition

For a simple graph $G=(V,E)$ of order $n>2$ and size m,

$$m = \frac{\sum_{v\in V} \#E(G-v)}{n-2}$$

where $\#E(G-v)$ is the number of edges in the graph $G-v$.

Proof

We start with the summation, and we will prove the summation is equal to $m(n-2)$:

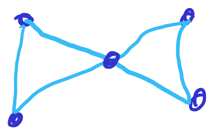
$$\sum_{v\in V} \#E(G-v) = \sum_{v\in V} \#E(G) - d_G(v) = \sum_{v\in V} \underbrace{\#E(G)}_{m} - \sum_{v\in V} d_G(v) = nm - 2m$$

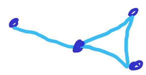
Conjecture (Reconstruction Conjecture – Kelly, Ulam, 1942)

If G is a simple graph with at least three vertices, then G is uniquely determined by the list of its vertex-deleted subgraphs (up to isomorphism).

Note that the hypothesis that G has at least three vertices is important. Otherwise, we would find a counterexample with two vertices, since both simple graphs with two vertices have the same set of vertex-deleted subgraphs. 

Example

 has vertex-deleted subgraphs $1 \times$ 

$4 \times$ 

To reconstruct the graph, we know that 4 vertices have degree $\#E(G)-4$ and 1 has degree $\#E(G)-2$. Using the proposition, the number of edges in G is $(2+4\times4)/3=6$. So the list of degrees is $(2,2,2,2,4)$, and the graph is connected.
That means that the vertices are in two cycles. The length of the cycles can be found by looking at the subgraphs: there is at least one cycle of length 3. Since the graph is simple, both cycles have length 3 and the graph has to me isomorphic to the bowtie.

Even though the conjecture is not proven, there are a number of cases that are known, and you will have to prove some of them as homework. Also, we can know some properties from the list of subgraph; for example if the graph is connected.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.3