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Graphic sequences

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We look at the list of degrees to get some information on the graph. We also look at what list of nonnegative integers can be the degree sequence of some graph.

Let G be a graph with vertices $v_1, v_2, \dots, v_n$. The <u>degree sequence</u> of G is the list $d(v_1), d(v_2), \dots, d(v_n)$. Usually, we write this sequence in decreasing order (and reorder the labels accordingly):

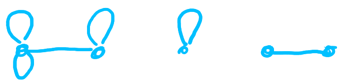
$$d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$$

Proposition
The nonnegative integers $d_1, d_2, \dots, d_n$ are the degree sequence of some graph if and only if their sum is even.

Proof
We need to prove that the condition is both necessary and sufficient.

=> (the condition is necessary) We already showed (on 04/10) that the sum of the degrees in a graph is always even.
<= (the condition is sufficient) This part of the prove is done by constructing a graph with a given degree sequence.
First, we consider all the vertices with odd degree (there is an even number of them). We pair them by drawing exactly one edge at each of these odd vertices. After this step, the number of endpoints to be added to every vertex is even, so we can add half this number of loops, making it a degree sequence.

Example
(5,3,2,1,1) can be realized on a (non-simple) graph in this way:

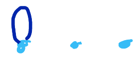


Of course, this technique does not work for simple graphs, because of the loops. Moreover, 5 cannot be the degree of a vertex in a simple graph with 5 vertices.

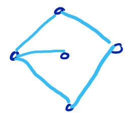
A graphic sequence is a list of nonnegative integers that is the degree sequence of some simple graph. A simple graph with degree sequence d realizes d.

Characterization of graphic sequences

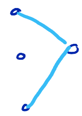
We already noticed the two obvious conditions for a nonnegative integers sequence to be graphic, i.e. the sum of degrees must be even and the maximal number cannot be greater than n−1. However, this is enough, as shown with the degree sequence (2,0,0), which must necessarily involve a loop.



Theorem (Havel 1955, Hakimi 1962)
The only one-element graphic sequence is (0).
For n>1, an integer list d of length n is graphic if and only if d' is graphic, where d' is obtained by deleting its largest element (Δ) and 1 from the Δ next largest degrees.

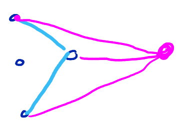


Example
The graph on the right has degree sequence d=(3,2,2,2,1).
It is obvisouly graphic by the picture. Here, Δ=3, and we obtain d' as (1,1,1,1). Notice that it is not the degree we obtain by deleting the highest-degree vertex (shown on the right), which would be (2,1,1,0). And (1,1,1,1) is also realizable, as shown below.

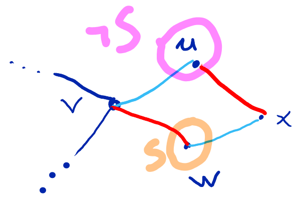


Proof (of theorem)
The case where there is only one vertex is obvious.
We need to prove that this condition is necessary and sufficient when n>1.
<= (sufficient) If d' is realizable, there exists a graph G' with vertices having d' as degrees. I want to add a vertex that has degree Δ greater than the largest degree of G'. To do so, I add the vertex and connect it to the Δ vertices with larger degrees in G', realizing d.

d' = (2,1,1,0)
d = (3,3,2,2,0)



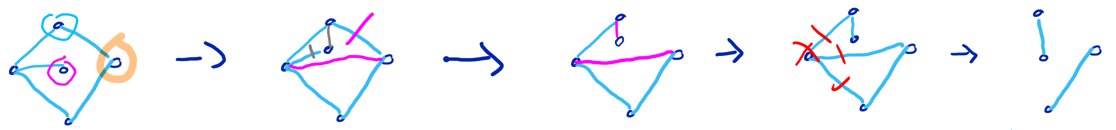
$\Rightarrow$ (necessary) There are two cases to consider. 1) the vertex v of degree Δ has neighbors that have the Δ next highest degree. Deleting v and its incident edges yield a graph with degree sequence d'.
2) Consider the neighborhood of v (the vertex of higher degree) and call it N. Let S be the set of the Δ vertices having the highest degree (except for v). Case 1) is when $N=S$, so here they are distinct. We will transform G to get $N=S$.
Take a vertex u in $N \setminus S$, so u is adjacent to v, but has a low degree, and take w in $S \setminus N$ (not adjacent to v, but high degree).
Since w has higher degree than u in $G \setminus \{v\}$, w has at least one neighbor x that is not adjacent to u.



No edge in the original graph.

By switching the edges uv and xw to vw and xu (from the blue to the red in the picture), we increase $|N \cap S|$. We repeat this process as long as $N \neq S$. When $N=S$, we use the first case.

Example



(3,2,2,2,1)

(1,1,1,1)

The case of loopless graphs

Multigraphs (even loopless) have a much easier characterization for degree sequences, as given by this theorem of Hakimi.

Theorem (Hakimi, 1962)
A sequence of decreasing nonnegative integers $d_1, d_2, \ldots, d_n$ is the degree sequence of a loopless graph if and only if its sum is even and

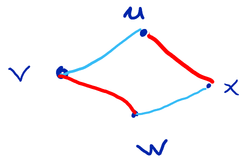
$$d_1 \leq d_2 + \ldots + d_n$$

Proof is left as homework for Monday's set.
Hint: You can proceed by construction, but it might be easier to do induction (not necessarily on the number of vertices).

Graphs with same graphic sequence

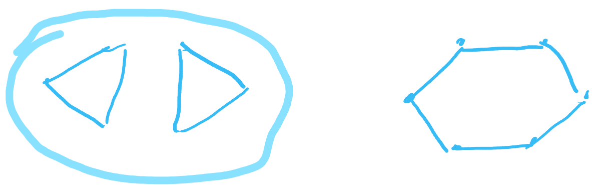
In the last proof, we exchanged the endpoints of some edges to get a new graph with the same graphic sequence.

A 2-switch is the replacement of a pair of edges $\{uv, wx\}$ by $\{ux, vw\}$, provided ux and vw did not already exist in the graph.

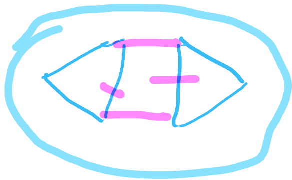


Remark
A 2-switch always preserves the degree of each vertex.

Example



They both have degree sequence $(2,2,2,2,2,2)$.



Theorem (Berge 1973)
Two simple graphs G and H have the same graphic sequence if and only if there is a sequence of 2-switches from G to H.

The proof is omitted, but can be found on page 47 of the textbook. The condition is clearly sufficient, as the 2-switches preserve the degree of each vertex.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.3