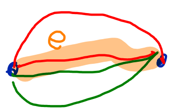
2.1.33 Let $G$ be an connected $n$-vertex graph. Prove that $G$ has exactly one cycle if and only if $G$ has $n$ edges.

Proof

=> Let $e$ be an edge from the only cycle.
By theorem from 4/6, $e$ is not a cut-edge. That means that $G-\{e\}$ is a connected graph with $n$ vertices.
Also, $G-\{e\}$ has no cycle, since $e$ belongs to the only cycle. Hence, $G-\{e\}$ is a tree, and has $n-1$ edges. That means that $G$ has $n$ edges.
<= If $G$ is connected and has $n$ edges, it must have at least one cycle (because it is not a tree). Let $e$ be an edge in a cycle.
$G-\{e\}$ is a tree, because it has $n$ vertices and $n-1$ edges; it is also connected because $e$ is not a cut-edge of $G$. That means that all the cycles of $G$ must pass through edge $e$.
By contradiction, assume there is at least two cycles, $c_1$ and $c_2$.
$e$ is an edge that belogs to both of them, so $c_1-\{e\}$ and $c_2-\{e\}$ are two paths with the same endpoints.

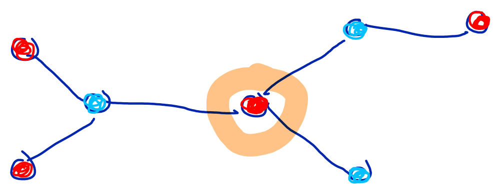



Gluing $c_1-\{e\}$ and $c_2-\{e\}$ yields a cycle that does not use the edge $e$. That is a contradiction, since all cycles of $G$ use $e$. The assumption that there were at least two cycles is wrong, and there is exactly one cycle.

2.1.29 Every tree is bipartite. Prove that every tree with at least two vertices has a leaf in its larger partite set (in both if they have equal size).

Proof 1: Let X and Y be the two independent sets, and assume that $|X| >= |Y|$.
Also, assume, by contradiction, that X has no leaf. That means that every vertex of X is incident to at least two edges that have an endpoint in Y:
$\#$ edges $\geq 2|X| \geq |X|+|Y| = \#$ vertices.
That contradicts the fact that the graph is a tree, since trees have fewer edges than vertices.
That proves that X must have at least one leaf.

Proof 2: by construction.



Start from the center of a tree, if it is a vertex. If the center is an edge, pick any of its enpoint. Call that vertex the root (highlighted in orange above). We can color it with red.
In the bipartition, all vertices at odd distance from the root are colored in blue, and all the vertices at even distance are red.

* Assume that there is no leaf in the red set. Then, we must prove that the blue set is larger.
Since there is no red leaf, every red vertex has at least one blue vertex that is further apart from the root and that is adjacent to that red vertex. Also, the root has at least two blue vertices as neighbors.
So for all the non-root red vertex, there is at least one blue vertex we can pair it with. One blue vertex can be matched with a red vertex by taking the only path from it to the root, and take the first (red) vertex of this path.

For every blue vertex, we paired it with one red vertex, but every red vertex is matched with at least one blue vertex, and at least two for the root. So there are more blue vertices than red vertices.

* The case of no blue leaf is analoguous to the case of no red leaf.