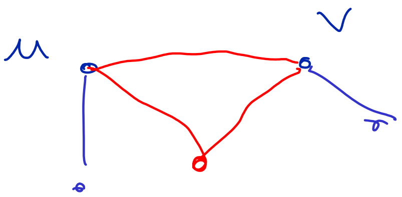
Proof of theorem 2.1.11

G has diameter at least 3.



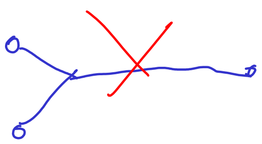
For every vertex w, there is at least one of either u or v that is not a neighbor of w.
WLOG, assume that uw is not an edge of G.
That means that uw is an edge of the complement of G, just as uv.
Whatever vertices {x,y} you pick in the complement of G, there is either
- a path xuy or xvy or an edge xy
- a path xuvy or yuvx

So there is a path of length at most 3 between any two vertices of the complement, making its diameter 3.

Problem 2.2.8

Count the sets of trees with n labeled vertices, giving two proofs for each: one using the Prüfer sequences, the other one by direct counting argument.

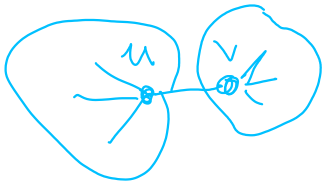
a) Trees that have 2 leaves

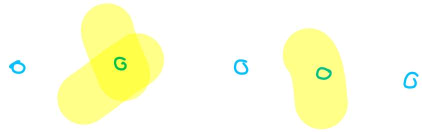


Observation: trees with 2 leaves are paths. So we can count paths.
Paths are in bijection with pairs made of one permutation and its reverse (the same permutation read from right to left). The total number of permutations is n!. This means that there are n!/2 paths of length n, and thus n!/2 trees with n vertices and 2 leaves.

Using Prüfer sequences:
Observation: Prüfer sequences of paths cannot have repetitions.
An element is added to the Prüfer sequences when a leaf adjacent to it is deleted. In paths, only one leaf can be deleted for every vertex, so there is no repeated element in the sequence. There are (n−2)! ways of ordering the elements of the sequence. We also need to choose the n−2 elements in the sequence among {1,2,…,n}. There are $\binom{n}{n-2} = \frac{n(n-1)}{2}$ ways of doing it.
Multiplying both values, there are (n−2)!(n−1)n/2=n!/2 such sequences.
This is also the number of trees with 2 leaves.

b) trees that have n−2 leaves





Step #1: We chose the 2 non−leaves vertices {u,v}. There are n(n−1)/2 choices.
Step #2: For every leaf, we choose among {u,v} the only neighbor of the leaf. There are 2^(n−2) such choices.
Step #3: Subtract the choices where all the leaves are around the same vertex; either they are all around u or around v, which means we should subtract 2 for each pair of non−leaves vertices.
The total number of such trees is thus $(2^{n-2}-2)\binom{n}{2}$

Proof using Prüfer sequences:
The Prüfer sequence of such a tree will only have the non−leaves labels in it. So it is a binary sequence. The number of binary sequences of length n−2 is 2^(n−2). There are n(n−1)/2 choices for the symbols that will occur in the binary sequence.
Also, we should not consider the case of all symbols in the sequence being the same. There are 2 ways of achieving it (either 000...0 or 11...1) for each choice of symbols.
So the total number of Prüfer sequences for this type of trees is $(2^{n-2}-2)\binom{n}{2}$