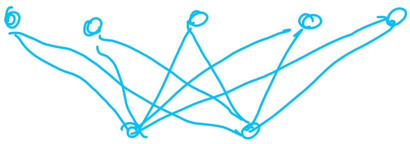
2.2.10 and 2.2.11

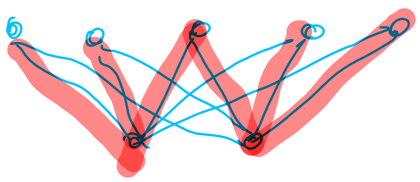
1. Compute the number of spanning trees of the complete bipartite graphs $K_{2,m}, K_{3,m}$

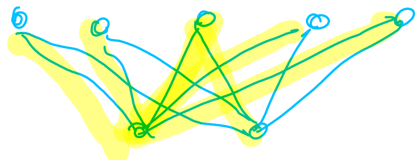
2. Compute the number of such trees up to isomorphisms

Start with $K_{2,m}$



Let X and Y be independent sets, with $|X|=2$.
Since the graph is bipartite, the only way of connecting the two vertices of X is to have a common neighbor in the tree, we need to pick that neighbor: there are m choices. Both edges incident to that vertex will be in the spanning tree.
For every other vertex, we need to choose exactly one neighbor (in X) to be in the spanning tree. There are 2 choices for each of them.
So there are $m2^{(m-1)}$ spanning trees.

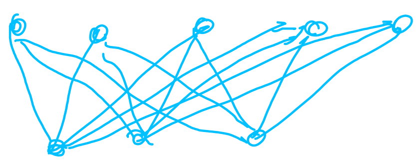


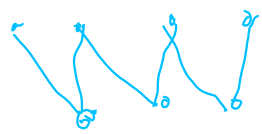


The spanning trees highlighted above cannot be the same up to isomorphism because they don't have the same degree sequence.

The number of isomorphism classes is $\lceil \frac{m}{2} \rceil$. This is obtained at the last step, when appending the $m-1$ remaining vertices of Y to a vertex of X, and is determined only by the size of the smaller neighborhood for a vertex in X, since vertices of X can be exchanged (with an isomorphism).

$K_{3,m}$





If there is a vertex of degree 3 in the spanning tree and in Y, we proceed in a smilar way, by chosing one vertex in Y that connects to all 3 vertices.

of X, and then, for every vertex in Y, we choose one incident edge. There are m3^(m−1) such spanning trees.

If there is no vertex of degree 3 in the spanning tree and in Y, then there are two vertices of degree 2 in the spanning tree and in Y. There are m choose 2 ways of chosing them; call them u and v. There will be exactly one vertex of X adjacent to both u and v, so there are 3 ways of chosing that vertex. There is also 2 ways of pairing the vertices u, v with the remaining vertices in X. For the m−2 remaining vertices of Y, we have 3^(m−2) choices for their unique neighbor. The total number of spanning trees with no vertex of degree 3 in it and in Y is $\binom{m}{2} \times 3 \times 2 \times 3^{m-2} = m(m-1)3^{m-1}$

The total number of spanning trees in K_{3m} is $m^2 3$^(m−1).