

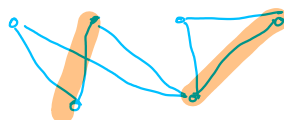
Math 38 – Graph Theory

Maximum and perfect matchings

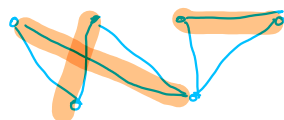
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In a graph, a matching is a subgraph with maximal degree 1 (so every vertex is connected to at most one other vertex).

A maximal, but not maximum matching



A maximum matching, also a perfect matching

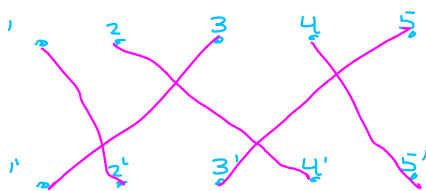


A vertex that appears in a matching is saturated, otherwise it is unsaturated.

A perfect matching in a graph is a matching that saturates every vertex.

Example

In the complete bipartite graph $K_{m,n}$, there exists perfect matchings only if $m=n$. In this case, the matchings of graph K_n represent bijections between two sets of size n . These are the permutations of n , so there are $n!$ matchings.



- Perfect matchings can only occur when the number of vertices is even.
- That is not a sufficient condition, as shown by the claw.



Example

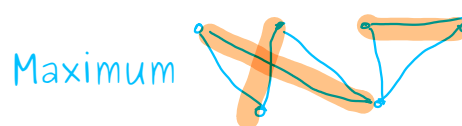
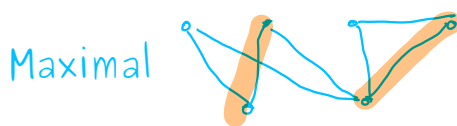
Counting the perfect matchings in a complete graph.

- K_n has no perfect matching if n is odd.
- Otherwise, it has $(n-1) \times (n-3) \times \dots \times 3 \times 1$ perfect matchings:
 - Label the vertices $1, \dots, n$
 - Match vertex 1 with any of its neighbors; there are $n-1$ possible choices
 - As long as there are still unsaturated vertices, match the smallest unsaturated vertex with another one. The number of ways to do so is $n-3$, then $n-5$, ..., until there is only one way to do so.

Maximum matchings

A matching of a graph is maximal if no edge can be added. It is maximum if no other matching of this graph has more edges than it.

Example



Perfect \Rightarrow Maximum \Rightarrow Maximal
In general, the converse is not true.

Can we transform a maximal graph into a maximum graph?



Let G be a graph and M be a matching of G . An M -alternating path is a path of G that alternates between edges in M and edges not in M . An M -augmenting path is an M -alternating path with both endpoints unsaturated.



Remark: When M is maximum, there is no augmenting path.

Theorem (Berge, 1957)

A matching M in a graph is a maximum matching if and only if the graph has no M -augmenting path.

Proof

\Rightarrow follows from the remark above

\Leftarrow We prove the converse: if it is not maximum, it has an augmenting path. If M is not maximum, then there is a matching M' with more edges.

We consider the subgraph H with edges that appear in exactly one of M and M' (not in both).

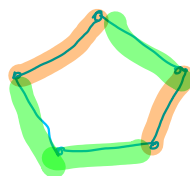
Claim: components of H are all either even cycles or paths.

- Even cycles have as many edges from M as from M' . This is because every endpoint can have at most one edge in M and one in M' .
- Since $|M'| > |M|$, there is at least one path in H with more edges of M' than edges of M . This is a path that starts and ends with unsaturated vertices of M , so this is an M -augmenting path.

Proof of the claim: That means that every vertex of H has degree at most 2, and that cycles have even length.

The maximal degree of H is 2 by construction. At most, one vertex can have one incident edge in M and one in M' .

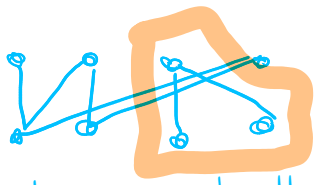
If a cycle has odd length, then most edges belong to the same matching, and there must be two edges belonging to the same matching and incident to the same vertex. That contradicts the construction of a matching.



Matchings in bipartite graphs

Example: Job assignments

If there are m jobs and n people, not all qualified for all the jobs, can we always fill all the jobs?



people
jobs

The edges are between a job and a qualified person for that job.

(The jobs cannot all be filled in this example).

Theorem (Hall's Theorem, 1935)

Let G be a bipartite graph with maximal independent sets X and Y . G has a matching that saturates every vertex of X if and only if the neighborhood of every $S \subseteq X$ has order at least $|S|$.

Consequence: Stable marriages. Watch the video.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 3.1