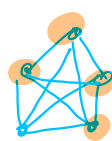
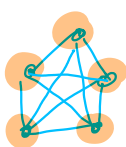


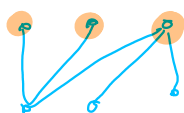
Recall from last class that a matching is a subset of edges such that no vertex appears twice as endpoints. We compare these notions with those of edge covers and vertex covers.

A vertex cover is a set  $S$  of vertices of  $G$  that contains at least one endpoint of every edge of  $G$ . The vertices in  $S$  cover  $G$ .

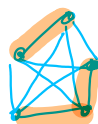
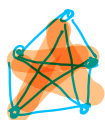
### Examples



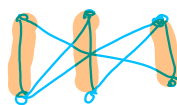
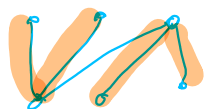
A cover of  $K_n$  has at least size  $n-1$ .



An edge cover is a set of edges of  $G$  that contains as endpoints every vertex of  $G$ .



The minimum number of edges in an edge cover is  $\lceil V/2 \rceil$ .



Sets	Items	Property
Matchings	Edges	At most one edge per vertex
Edge covers	Edges	At least one edge per vertex
Vertex covers	Vertices	At least one vertex per edge
Independent set	Vertices	At most one vertex per edge

When are:

Vertex covers and independent sets equal?

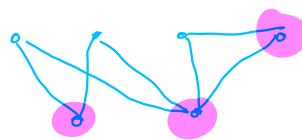
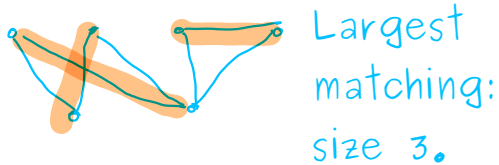
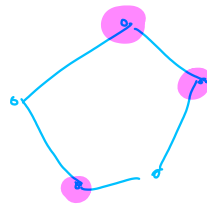
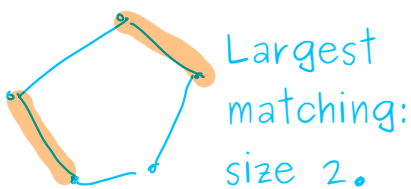
Matchings and edge covers equal?

## Matchings and vertex covers

From the table above, it might seem that matchings and vertex covers are not related. However, consider a matching. In a vertex cover, every edge of the matching has to be covered by one of its endpoint. So the edge  $uv$  has either  $u$  or  $v$  (or both) in the vertex cover. Also,  $u$  (and  $v$ ) cannot belong to more than one edge in the matching. So we have the following proposition:

### Proposition

If  $M$  is a matching of  $G$  and  $S$  is a vertex cover of the same graph  $G$ ,  $|M| \leq |S|$ .



In the last example the matching and the vertex cover have the same size ( $|M| = |S|$ ). Of course, one can get smaller matchings and larger vertex covers, but the first example shows it is not possible to get a matching and a vertex cover of the same size.

### Remark

The statements

"For any matching  $M$  and any vertex cover  $S$ ,  $|M| \leq |S|$ "

and

"The maximum size of a matching is always at most the minimum size of a cover"

are equivalent.

Theorem (König, Egerváry, 1931, independently)

If  $G$  is a bipartite graph, then the maximum size of a matching in  $G$  is equal to the minimum size of a vertex cover in  $G$ .

Proof

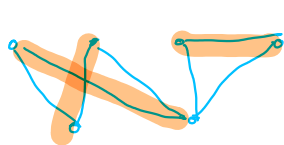
As an assignment, read either

- the proof of Theorem 3.1.16 in the textbook.
- the short proof of Romeo Rizzi (the paper is on Canvas).

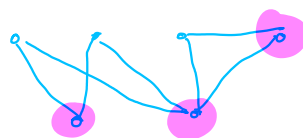
The first one is a proof by construction, the second one is a proof by contradiction.

Remark

This is not an if and only if statement. For example, the graph below contains a triangle, but has a matching a vertex cover of the same size.

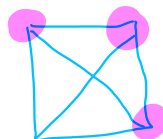
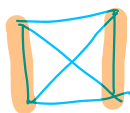


Largest  
matching:  
size 3.



Smallest vertex cover:  
size 3.

Also, being a perfect matching is not enough, as shown by this example with just 4 vertices.



Complete graphs have vertex covers of size  $n-1$  and maximum matching of size  $\lfloor \frac{n}{2} \rfloor$

Remark

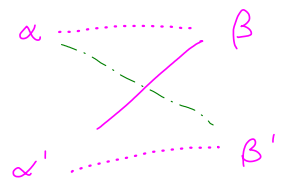
The theorem above is an example of a min-max relation: that means that solving an optimization problem calling for the minimum of something in a graph is equivalent to solving an optimization problem for the maximum. Here: the maximum matching and the minimum vertex cover. We will see more of them over the term.

# Optimization problems

Sets	Items	Problem	Notation
Matchings	Edges	Finding maximum	$\alpha'(G)$
Edge covers	Edges	Finding minimum	$\beta'(G)$
Vertex covers	Vertices	Finding minimum	$\beta(G)$
Independent set	Vertices	Finding maximum*	$\alpha(G)$

\* This is defined as the independence number.

As an example on how to use these notations, König-Egerváry theorem states that  $\alpha'(G) = \beta(G)$  for bipartite graphs. For any graph  $G$ ,  $\alpha'(G) \leq \beta(G)$ .



## Lemma ( $\alpha - \beta$ )

In a graph  $G=(V,E)$ ,  $S$  is an independent set if and only if  $V-S$  is a vertex cover. Hence,  $|V| = \alpha + \beta$ .

## Proof

Let  $S$  be an independent set, meaning there is no edge between two vertices of  $S$ ; every edge has at least one endpoint in  $V-S$ . That means that  $V-S$  is a vertex cover.

Conversely, if  $S'$  covers  $G$ , every edge has at least one endpoint in  $S'$ , so  $V-S'$  is independent.



Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 3.1