Math 38 — Graph Theory

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Vertex cut, connectivity, covers

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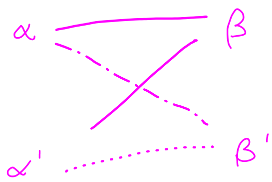
Recall from last lecture:

Sets	Items	Problem	Notation
Matchings	Edges	Finding maximum	$\alpha'(G)$
Edge covers	Edges	Finding minimum	$\beta'(G)$
Vertex covers	Vertices	Finding minimum	$\beta(G)$
Independent set	Vertices	Finding maximum	$\alpha(G)$



We still need to find some relations between α' and β', and beteween α and β' (which will be possible under some conditions only).

Theorem (Gallai, 1959, $\alpha'-\beta'$)
If G is a graph without isolated vertices and n vertices in total, then $\alpha'(G)+\beta'(G)=n$. (max. matching + min edge cover)

Proof
2 steps:

1. From a maximum matching (of size $\alpha'(G)$), construct a minimum cover of size $n-\alpha'(G)$. That implies that $n-\alpha'(G)\geq\beta'(G)$.
2. From a minimum cover (of size $\beta'(G)$), construct a matching of size $n-\beta'(G)$. That implies that $\alpha'(G)\geq n-\beta'(G)$.

These two steps prove that $\alpha'(G)+\beta'(G)=n$.

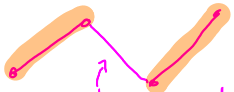
Step 1: Let M be a maximum matching, and let S be a set of vertices. At first, $S=M$. S will be, at the end of the process, an edge cover. For every unsaturated vertex, add to S an edge incident to it.
Claim: S is a cover of size $n-|M|$.
- S is a cover, because edges of M cover saturated edges, and we

added edges to cover unsaturated edges.

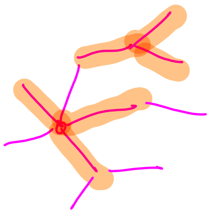
- S has size $n-|M|$: we know S has size $|M|+$"number of unsaturated vertices". However, the number of unsaturated vertices is $n-2|M|$, because every edge of the matching saturates 2 vertices, with no edges saturating the same vertex. So $|S| = |M|+n-2|M| = n-|M|$.

Let S be a minimum cover. We want to construct a matching of size $n-|S|$.

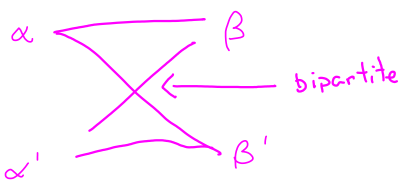
Observation: In a minimum cover, there is no path of length 3.



Since there is no path of length 3 in a cover, the components of a cover are all stars. Assume there are k components. Then we can create a matching of size k, by taking one edge in every star.
To find the value of k, we count the number of centers of the star. For each edge of the cover, there is exactly one non-center (of a star). So the number of stars is $n-\#$"non-centers"$=n-|S|$. So there is a matching of size $n-|S|$.



Corollary (König, 1916, $\alpha-\beta'$)
If G is a bipartite graph with no isolated vertex, then the size of a maximum independent set is the size of a minimum edge cover.



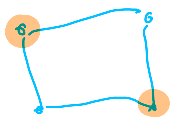
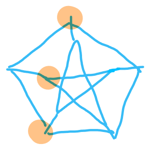
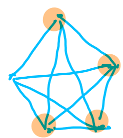
Cuts and connectivity

A vertex cut (or separating set) is a subset of vertices S such that $G-S$ has more than one component.

The connectivity of G, $\kappa(G)$, is the minimum size of a separating set, if it exists, or $n-1$.

A graph is k-connected if its connectivity is at least k.

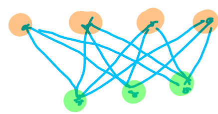
Examples

Disconnected = connectivity 0

Connected = 1-connected

Cycles of length at least 3 have connectivity 2



Petersen graph has connectivity 3.

Complete graph K_n has connectivity $n-1$.

Complete bipartite graph $K_{m,n}$ has connectivity $\min\{n,m\}$.

By convention, we say the graph with one vertex has connectivity 0.

Proposition

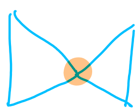
The connectivity of a connected graph is at most its minimum degree.

Proof

One can isolate a single vertex removing all the vertices around it.

Remark

The connectivity of a connected graph is not at least its minimum degree.



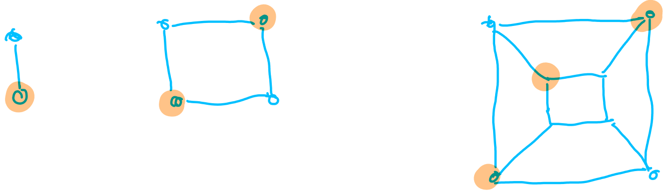
Minimum degree 2, but there is a cut-vertex => connectivity 1.

Example

The hypercube H_k has connectivity k.

Of course, since it is k-regular, it has connectivity at most k.

We can prove by induction it has connectivity at least k:



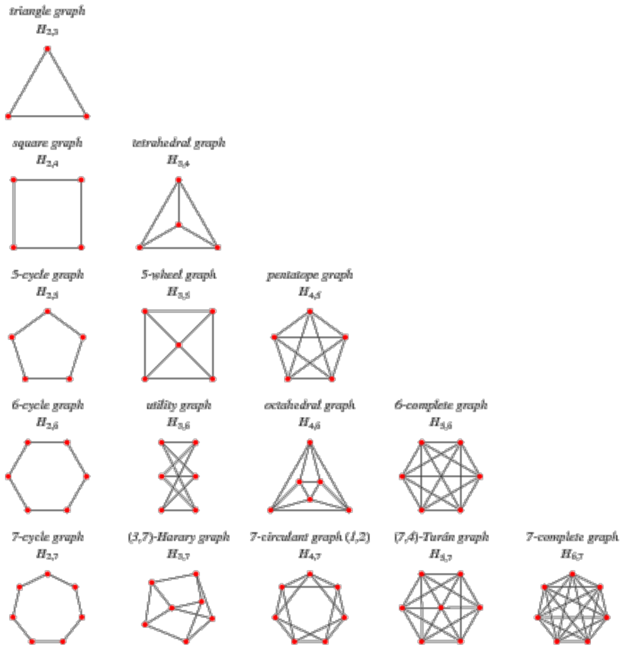
Example: Harary graphs

Harary graphs $H_{k,n}$ are graphs with n vertices and $\lceil \frac{nk}{2} \rceil$ edges, being as regular as possible.

They have connectivity k:

- k is the minimum degree of $H_{n,k}$

Homework: Read the proof in the text-book that it has connectivity at least k.



Corollary (Harary, 1962)

The minimum number of edges in a k-connected graph with n vertices is $\lceil \frac{nk}{2} \rceil$

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001.
Section 3.1 (covers) and section 4.1 (connectivity)