Math 38 — Graph Theory
Network and flows

Nadia Lafrenière
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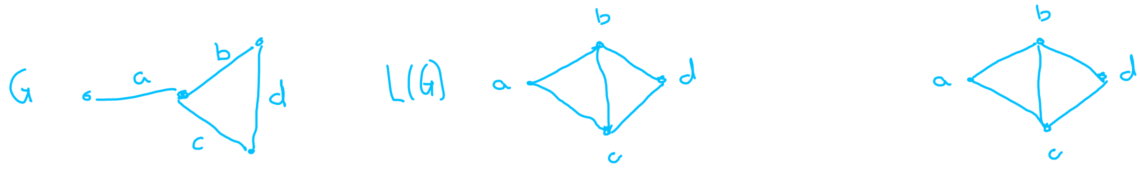
We progress in our journey to analyzing flow in a network. We first introduce line graphs (and digraphs) to express dual problems, and then move on to networks, flows and capacity.

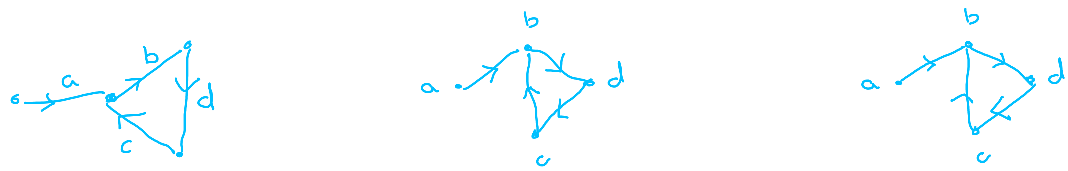
Line graphs

Goal: Introduce a way to translate edge Menger's theorem and other results on paths in terms of edges.

Let $G=(V,E)$ be a graph. Its line graph $L(G)$ has vertices E, and edges of $L(G)$ exist for two edges of G (vertices of $L(G)$) if they are incident to the same vertex in G.



The same can be done with digraphs. In this case, there is a directed edge from e in E to f if there is a path in $D=(V,E)$ from e to f.



Remark
G is disconnected whenever $L(G)$ is.

Theorem
If u and v are distinct vertices in a graph (or digraph) G, then the minimum size of an uv-disconnecting set (of edges) equals the maximum size of pairwise edge-disjoint uv-paths.

Sketch of the proof
Use Menger's theorem with $L(G)$.

Corollary

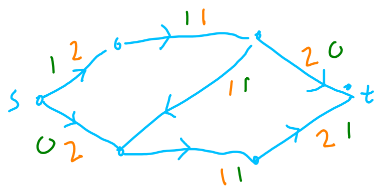
The edge-connectivity of a graph (or a digraph) is the maximum number k such that there is at least k edge-disjoint uv-paths for all pairs of vertices {u,v}.

Maximum Network Flow

A network is a directed graph with a nonnegative capacity c(e) on each edge e. A network has distinguished vertices: a source s and a sink t.

A flow f in a network D assings a value f(e) to edge e. For vertices, we write $f^+(v)$ for the total flow of the edges leaving v and $f^-(v)$ for the flow entering v.

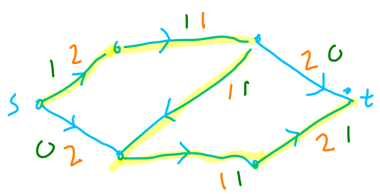
A flow is feasible if
- $0 \le f(e) \le c(e)$ for every edge e. Capacity constraint
- $f^+(v) = f^-(v)$ for every vertex except source and sink. Conservation constraint



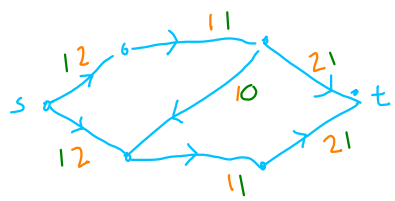
- capacity
- flow

The value of a flow is the net flow of the sink $(f^-(t) - f^+(t))$.
A maximum flow is a feasible flow of maximum value.

Maximal flow



Value 1



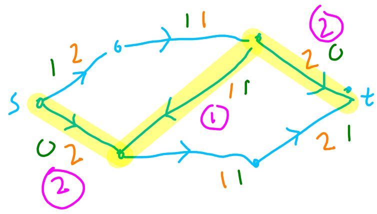
Value 2

Maximum flow

To increase the value of a maximal, but not maximum flow, we use f-augmenting paths. P is an f-augmenting path if
- it is going from source to sink.
- when P follows e in the forward direction, $f(e) < c(e)$.

Let $\varepsilon(e) = c(e) - f(e)$.
- when P follows e in the backward direction, $f(e) > 0$. Let $\varepsilon(e) = f(e)$.

The tolerance of P is the minimum value of $\varepsilon(e)$ over edges in P.



Tolerance 1

Lemma

If P is an f-augmenting path with tolerance z, then we can create a flow f' with value $\text{value}(f)+z$ in the following way:

- if e not in P, $f'(e)=f(e)$
- if e is forward in P, $f'(e)=f(e)+z$
- if e is backward in P, $f'(e)=f(e)-z$.

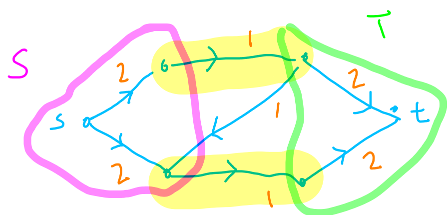
Proof

We must prove that f' is a flow (capacity and conservation constraints) and that the result has value z higher than the value of f.

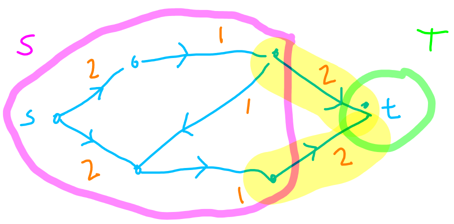
(Proof in the video)

Source/sink cut

Given a partition of the vertices in a network D with source s and sink t, consider a partition of the vertices of D into a source set S (containing s) and a sink set T (with t). A source/sink cut is an edge cut $[S,T]$. Its capacity, $\text{cap}(S,T)$ is the total capacity of the edges from S to T.



Capacity 2



Capacity 4

Teaser for next class...

Theorem (Max-flow Min-cut, Ford-Fulkerson, 1956)

The maximum flow in a network is the minimum capacity of a source/sink cut.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Sections 4.2, 4.3