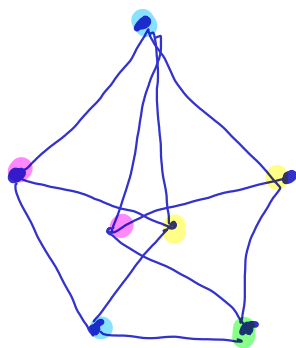


5.1.1 Find the chromatic number, the clique number and the independence number of the graph below.



The clique number is at least 3, since there are triangles. Since  $K_4$  is not a subgraph, the clique number is 3.

The independence number is at least 2 (since this is not the complete graph). It is at most 2 for the following reason:

— Assume there is an independent set of size 3. If it contains the top vertex (the one of degree 4), then it must have the two vertices at the bottom, but they are adjacent. So the top vertex is not in the independent set.

— Once we removed the top vertex, we are left with two triangles, joined by one edge. No two vertices in the same triangle can be in the same independent set, so it has size at most 2.

That proves that the independence number is 2.

We know the chromatic number is at least 3 (because of the triangles) and at most 4 (because of the coloring above).

What if the chromatic number is 3?

Then, recall that proposition from today's lecture: The chromatic number satisfies  $\chi(G)\alpha(G) \geq |V|$ .

That means that the chromatic number has to be at least 4.  
So the chromatic number of this graph is 4.

5.1.5 (a) Let  $G$  be a graph with components  $H$  and  $J$ . Describe the chromatic number of  $G$  in terms of those of  $H$  and  $J$ .

$$\chi(G) = \max \{\chi(H), \chi(J)\}$$

Assume, wlog, that  $H$  has the larger chromatic number among  $H$  and  $J$ .

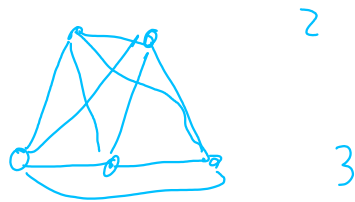
Then,  $G$  must be colored by at least  $\chi(H)$  colors, because  $\chi(H)$  colors are needed to color the vertices of  $H$ .

Also,  $\chi(H)$  colors are enough because the vertices of  $J$  can be colored with the same colors. This is because, there is no adjacency relationships between vertices of  $H$  and  $J$ .

(b) Let  $G$  be a graph made in the following way:  $H$  and  $J$  are two components.  $G$  is the join of  $H$  and  $J$ , that is the edges inside  $H$  and  $J$  are the same as in the original graphs, and for every pair of vertices  $h$  in  $H$  and  $j$  in  $J$ ,  $hj$  is an edge of  $G$ .

What is the chromatic number of  $G$  in terms of those of  $H$  and  $J$ ?

Example



The chromatic number of  $G$  is  $\chi(H) + \chi(J)$ .

$\chi(H)$  colors are needed to color  $H$ , and  $\chi(J)$  colors are needed for  $J$ .

Also, the same colors cannot be used for vertices in  $H$  and  $J$ :

otherwise, there are vertices  $j$  in  $J$  and  $h$  in  $H$  that are of the same color, but they are adjacent in  $G$ , so the coloring is not a proper coloring. Therefore, the number of colors needed is  $\chi(H) + \chi(J)$ .